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A SHORT COURSE

OF

ELEMENTARY PLANE TRIGONOMETRY

BY

CHARLES PENDLEBURY, M.A.

SENIOR MATHEMATICAL MASTER OF ST PAUL'S SCHOOL; F.R.A.S.;
FORMERLY SCHOLAR OF ST JOHN'S COLLEGE, CAMBRIDGE.

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PREFACE.

THIS Short Course of Elementary Plane Trigonometry is intended for the use of Junior Students and of those others who do not require so complete a knowledge of the subject as would be obtained from a study of my larger work. In its range this book will be found sufficient for Candidates for the Oxford and Cambridge Local Examinations, the College of Preceptors, the Cambridge Previous Examination, the London Intermediate Examination in Arts, the Science and Art Examinations, College Entrance Examinations, and other examinations of the same character.

C. P.

ST PAUL'S SCHOOL, LONDON,

December, 1899.

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19 1

FORMULÆ.

ARTICLE

$$13. \pi = 3.14159265\dots = \frac{355}{113} \text{ (nearly)} = \frac{22}{7} \text{ (nearly).}$$

$$13. \text{ Circumference of a circle} = \pi d = 2\pi r.$$

$$16. 1 \text{ radian} = 57^\circ 17' 45'', \text{ nearly.}$$

$$17. 180^\circ = 200^{\text{gr}} = \pi \text{ radians.}$$

$$19. \frac{D}{180} = \frac{G}{200} = \frac{\theta}{\pi}.$$

$$11, 22. \text{ Circular measure} = n^\circ. \text{ of radians} = \frac{\text{arc subtended}}{\text{radius}}.$$

$$23. \text{ Arc of a circle} = r\theta.$$

$$31. 45^\circ; \text{—picture a right-angled triangle whose sides are } 1, 1, \sqrt{2};$$

$$\sin 45^\circ = \frac{1}{\sqrt{2}}; \quad \cos 45^\circ = \frac{1}{\sqrt{2}}; \quad \tan 45^\circ = 1.$$

$$60^\circ \text{ and } 30^\circ; \text{—picture a rt-angled triangle whose sides are } 1, \sqrt{3}, 2;$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2}, \quad \cos 60^\circ = \frac{1}{2}, \quad \tan 60^\circ = \sqrt{3};$$

$$\sin 30^\circ = \frac{1}{2}, \quad \cos 30^\circ = \frac{\sqrt{3}}{2}, \quad \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

$$34. 0^\circ; \text{ perpendicular vanishes; intercept coincides with hypotenuse;}$$

$$\sin 0^\circ = 0, \quad \cos 0^\circ = 1, \quad \tan 0^\circ = 0.$$

$$90^\circ; \text{ intercept vanishes; perpendicular coincides with hypotenuse;}$$

$$\sin 90^\circ = 1, \quad \cos 90^\circ = 0, \quad \tan 90^\circ = \infty.$$

For geometrical problems.

$$\sqrt{2} = 1.4142; \quad \sqrt{3} = 1.73205; \quad \sqrt{5} = 2.23607; \quad \sqrt{6} = 2.44949.$$

$$46. \quad (i) \quad \tan A = \frac{\sin A}{\cos A};$$

$$(ii) \quad \sin^2 A + \cos^2 A = 1;$$

$$(iii) \quad 1 + \tan^2 A = \sec^2 A;$$

$$(iv) \quad 1 + \cot^2 A = \operatorname{cosec}^2 A.$$

$$49. \quad \sin(A+B) = \sin A \cos B + \cos A \sin B,$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B.$$

$$50. \quad \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$51. \quad \sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}, \quad \cos 75^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}, \quad \tan 75^\circ = 2 + \sqrt{3}.$$

ARTICLE

52. $\sin(A - B) = \sin A \cos B - \cos A \sin B$,
 $\cos(A - B) = \cos A \cos B + \sin A \sin B$.
53. $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.
54. $\sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}$, $\cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}$, $\tan 15^\circ = 2 - \sqrt{3}$.
56. $\sin P + \sin Q = 2 \sin \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$,
 $\sin P - \sin Q = 2 \cos \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$,
 $\cos P + \cos Q = 2 \cos \frac{1}{2}(P + Q) \cos \frac{1}{2}(P - Q)$,
 $\cos Q - \cos P = 2 \sin \frac{1}{2}(P + Q) \sin \frac{1}{2}(P - Q)$.
57. $2 \sin A \cos B = \sin(A + B) + \sin(A - B)$,
 $2 \cos A \sin B = \sin(A + B) - \sin(A - B)$,
 $2 \cos A \cos B = \cos(A + B) + \cos(A - B)$,
 $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$.
59. $\sin 2A = 2 \sin A \cos A$,
 $\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A$,
 $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$.
60. $\left. \begin{aligned} 1 - \cos A &= 2 \sin^2 \frac{A}{2} \\ 1 + \cos A &= 2 \cos^2 \frac{A}{2} \end{aligned} \right\} \text{whence } \tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A}$.
61. $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$.
62. $\sin 3A = 3 \sin A - 4 \sin^3 A$,
 $\cos 3A = 4 \cos^3 A - 3 \cos A$,
 $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$.
63. $\sin 18^\circ = \frac{\sqrt{5} - 1}{4}$.
74. **180°**; perpendicular vanishes; intercept coincides with hypotenuse, and is negative;
 $\sin 180^\circ = 0$, $\cos 180^\circ = -1$, $\tan 180^\circ = 0$.
75. **270°**; intercept vanishes; perpendicular coincides with hypotenuse, and is negative;
 $\sin 270^\circ = -1$, $\cos 270^\circ = 0$, $\tan 270^\circ = -\infty$.
- 79-82. **180° - A**, **180° + A**, **360° - A**, **-A**; trigonometrical ratios are numerically equal to corresponding ratios of A; algebraic sign is determined by quadrant.
- 85-87. **90° - A**, **90° + A**; trigonometrical ratios are numerically equal to corresponding co-ratios of A; algebraic sign is determined by quadrant.

ARTICLE

95. $\log_a 1 = 0$; $\log_a a = 1$; whatever a may be,

$$\log_a (mn) = \log_a m + \log_a n,$$

$$\log_a \left(\frac{m}{n} \right) = \log_a m - \log_a n,$$

$$\log_a (m^r) = r \log_a m.$$

96. $\log_b a \times \log_a b = 1$.

118. If $A + B + C = 180^\circ$,

$$(i) \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

$$(ii) \quad \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2};$$

$$(iii) \quad \cos A + \cos B + \cos C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1.$$

123-132. Each of the following formulæ is one of a set of three:—

$$(i) \quad a = b \cos C + c \cos B. \quad (ii) \quad a^2 = b^2 + c^2 - 2bc \cos A.$$

$$(iii) \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}. \quad (iv) \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

$$(v) \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}. \quad (vi) \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$(vii) \quad \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

$$(viii) \quad \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}. \quad (ix) \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

145-6. Area of triangle $= \frac{1}{2} bc \sin A = \frac{1}{2} ac \sin B = \frac{1}{2} ab \sin C$

$$= \sqrt{s(s-a)(s-b)(s-c)}.$$

$$152. \quad R = \frac{a}{2 \sin A} = \frac{b}{2 \sin B} = \frac{c}{2 \sin C} = \frac{abc}{4\Delta}.$$

$$154. \quad r = \frac{a}{\cot \frac{B}{2} + \cot \frac{C}{2}} = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}} = \frac{\Delta}{s}.$$

$$156. \quad r_1 = \frac{a}{\tan \frac{B}{2} + \tan \frac{C}{2}} = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}} = \frac{\Delta}{s-a}.$$

160. Pedal Triangle:—angles are $180^\circ - 2A$, $180^\circ - 2B$, $180^\circ - 2C$;
sides are $a \cos A$, $b \cos B$, $c \cos C$.

$$162. \quad \tan \theta = \tan (n\pi + \theta),$$

$$\cot \theta = \cot (n\pi + \theta);$$

$$164. \quad \cos \theta = \cos (2n\pi \pm \theta),$$

$$\sec \theta = \sec (2n\pi \pm \theta);$$

$$165. \quad \sin \theta = \sin \{n\pi + (-1)^n \theta\};$$

$$\operatorname{cosec} \theta = \operatorname{cosec} \{n\pi + (-1)^n \theta\};$$

where n is any integer, positive or negative.

A SHORT COURSE OF ELEMENTARY PLANE TRIGONOMETRY.

PART I.

I. AN ANGLE.

1. In Plane Trigonometry we investigate the properties of plane rectilinear angles, and of such plane figures as are dealt with in Euclid. The Triangle is of course the most important of these figures.

2. In no case considered by Euclid does the magnitude of an angle equal, or exceed, two right angles, but it is not necessarily subject to any such limitation. For the purposes of our present subject, we will adopt the following definition.

DEFINITION. If a straight line OA have one end O fixed, and be made to rotate about that end from the position OA to another position such as OP , it is said to describe an angle AOP .

An angle is *measured* by the extent of the line's rotation.

3. When the line has rotated from OA to OB , as in the figure, it has described a right angle, and when it has reached OC , it has described two right angles.

But the rotating line need not stop at OC . It may go on until it arrives back again at OA , when it will have described 4 right angles, and it may then continue to rotate on and on as many times round O as we please.

Thus the angle bounded by OA and OP may be simply the *visible* angle AOP, or it may be

the visible angle AOP + any integral multiple of 4 right angles;
but the student will find that the trigonometrical properties are in all cases the same as those of the *visible* angle AOP.

4. The **measure** of an angle is the *number* which expresses how many times it contains the *unit* angle.

5. A **unit angle** is one which serves the same purpose in measuring angles that a *foot*, or a *yard*, serves in reckoning the distance between one place and another.

Rectangular measure.

6. In Euclid the magnitude of an angle is expressed with reference to a **right angle**, which is Euclid's *standard* or *unit of angular measurement*. This is one of the units adopted also in Trigonometry, and the measure of an angle is then called its **rectangular measure**.

A right angle is divided and subdivided into degrees, minutes and seconds, the relations between which are stated in the following table:—

SEXAGESIMAL MEASURE.

1 right angle = 90 degrees (90°).

1 degree = 60 minutes ($60'$).

1 minute = 60 seconds ($60''$).

The measure of an angle containing 523 *degrees*, 15 *minutes* and 49 *seconds* is written thus:— $523^\circ 15' 49''$. It may be noticed that although a right angle is here the unit of measurement, it is not mentioned by name.

The above method of dividing and subdividing a right angle is called the **sexagesimal** method. It is the only one that is, or ever has been, in common use.

7. Certain eminent French mathematicians, towards the end of the eighteenth century, wishing to have a decimal system, suggested that the right angle should be divided and subdivided as follows:—

CENTESIMAL MEASURE.

1 right angle = 100 grades (100^{gr}).

1 grade = 100 minutes ($100'$).

1 minute = 100 seconds ($100''$).

The measure of an angle containing 523 grades, 15 minutes and 49 seconds would be written thus:—523^{gr} 15' 49". This is called the **French**, or **centesimal**, method.

8. The **advantages** of the *centesimal* method would be that the measure of an angle of 52^{gr} 5' 39", for example, could be expressed *at sight* in any of the following forms:—52^{gr} 5' 39, 52·0539 grades, ·520539 of a right angle, 5205·39 centesimal minutes, and 520539 centesimal seconds. The process of conversion or *reduction* would thus consist of nothing more than the insertion of a decimal point in its proper place, or its transference from one place to another. Practical difficulties have however prevented the adoption of the centesimal system even in France.

9. An angle expressed *sexagesimally* as 52° 5' 39" cannot be reduced *at sight* to the decimal of a right angle. We must use the method explained in *Arithmetic*, Art. 192.

$$\begin{array}{r|l} 60 & 39 \text{ seconds} \\ 60 & \overline{5 \cdot 65} \text{ minutes} \\ 90 & \overline{52 \cdot 0941\dot{6}} \text{ degrees} \end{array}$$

Hence $52^\circ 5' 39'' = \cdot 5788240\dot{7}$ of a right angle.

10. For examination purposes it is necessary for the student to be able to convert the measure of an angle from sexagesimal measure to centesimal, and *vice versa*.

To convert from sexagesimal measure to centesimal, we first express the angle as the decimal of a right angle, by the method of Art. 9. Thus we found that

$$52^\circ 5' 39'' = \cdot 5788240\dot{7} \text{ of a right angle;}$$

$$\text{hence } 52^\circ 5' 39'' = 57^{\text{gr}} 88' 24'' \cdot 074.$$

Conversely, to convert from centesimal measure to sexagesimal, we express the angle as the decimal of a right angle, and then by the process explained in *Arithmetic*, Art. 193, we reduce the decimal of a right angle to degrees, minutes and seconds.

For example, $52^{\text{gr}} 5' 9'' = \cdot 520509$ of a right angle;

$$\begin{array}{r|l} 90 & \\ \hline 46 \cdot 845810 & \\ 60 & \therefore 52^{\text{gr}} 5' 9'' = \cdot 520509 \text{ of a rt. angle} \\ \hline 50 \cdot 748600 & = 46^\circ 50' 44'' \cdot 916. \\ 60 & \\ \hline 44 \cdot 916000 & \end{array}$$

EXAMPLES 1.

Express the following angles in centesimal measure:—

(1) 45° .	(5) $36^\circ 54'$.	(9) $2^\circ 48' 45''$.
(2) 15° .	(6) $56^\circ 45' 30''$.	(10) $37^\circ 0' 12''$.
(3) 12° .	(7) $5^\circ 37' 30''$.	(11) $12' 36''$.
(4) $22^\circ 30'$.	(8) $54^\circ 0' 24''$.	(12) $8''$.

Express the following angles in sexagesimal measure:—

(13) $26^{\text{gr}} 50'$.	(16) $13^{\text{gr}} 41' 9''$.	(19) $48^{\text{gr}} 0' 50''$.
(14) $42^{\text{gr}} 25'$.	(17) $82^{\text{gr}} 3' 5''$.	(20) $2' 54''$.
(15) $64^{\text{gr}} 18' 36''$.	(18) $2^{\text{gr}} 8' 7''$.	(21) $5''$.

Circular Measure.

11. The special unit of angular measurement which is used in Trigonometry, but not in Euclid, is the angle between two radii of a circle which intercept on the circumference an arc equal in length to the radius.

DEFINITION. An angle subtended at the centre of a circle by an arc equal in length to the radius is called a **radian**. The **circular measure** of an angle is the *number* of radians contained in it.

A quantity cannot be used as a *unit* of measurement unless it is of constant magnitude. We must therefore prove that a *radian* is an angle of unvarying size whatever the radius of the circle may be. We will first prove that the circumference of a circle is always the same multiple of the radius.

12. PROPOSITION I. *The circumference of a circle is always the same multiple of the radius.*

Let there be two circles with a common centre O ; and let $P, Q, R, S, \dots$ be equidistant points on the outer circumference, and $A, B, C, D, \dots$ the points in which $OP, OQ, OR, OS, \dots$ respectively meet the inner circle.

Join $PQ, QR, RS, \dots$, and $AB, BC, CD, \dots$

Then, by Euclid VI. 4,

$$\begin{aligned} \frac{OP}{OA} &= \frac{PQ}{AB} = \frac{QR}{BC} = \frac{RS}{CD} = \dots \\ &= \frac{PQ + QR + RS + \dots}{AB + BC + CD + \dots} \end{aligned}$$

If now the distances PQ, QR, RS, ... be diminished indefinitely, and consequently the number of points P, Q, R, S, ... be increased indefinitely, the polygon of which PQ, QR, RS, ... are sides becomes indistinguishable from the outer circle itself, and the polygon of which AB, BC, CD, ... are sides becomes indistinguishable from the inner circle.

Hence, when the number of points is increased indefinitely, and the polygon becomes coincident with the circle, we see that

$$\frac{OP}{OA} = \frac{\text{circumference of outer circle}}{\text{circumference of inner circle}},$$

$$\text{i.e., } \frac{\text{circumference of outer circle}}{OP} = \frac{\text{circumference of inner circle}}{OA}.$$

Hence we may infer that the ratio which the circumference of a circle bears to the radius, and consequently the ratio which it bears to the diameter, does not vary as we go from circle to circle.

13. The exact numerical value of the ratio of circumference to diameter cannot be ascertained, but it may be represented by 3.14159265.... This can be expressed approximately by $\frac{22}{7}$, and more nearly by $\frac{355}{113}$ and 3.1416. Of these, the value $\frac{22}{7}$, or $3\frac{1}{7}$, is generally used when great accuracy is not required.

The *exact* numerical value of the ratio of circumference to diameter is denoted by the symbol π . Thus,

$$\text{circumference of circle} = \pi \times \text{diameter} = 2\pi \times \text{radius}.$$

EXAMPLES 2.

Assuming that $\pi = 3\frac{1}{7}$, find the length of the circumference of a circle the length of whose radius is

(1) 2 in.	(3) $3\frac{1}{2}$ ft.	(5) 4 ft. 9 in.
(2) 5 ft.	(4) 6 ft. 3 in.	(6) 3 yds. 1 ft. 6 in.

Find the length of the radius, when the circumference is

(7) 8 in.	(9) .25 in.	(11) 1 ft. 6 in.
(8) 500 yds.	(10) $2\frac{4}{5}$ ft.	(12) 5 m. 3 fur.

14. It should be noticed here that a right angle, or an angle of 90° , at the centre of a circle, is subtended by an arc which is equal to $\frac{1}{4}$ of the circumference and is called a **quadrant**; that two right angles, or an angle of 180° , are subtended by an arc equal to *half* the circumference; and that an angle of 360° , or four right angles, is subtended by an arc equal to the *whole* circumference of the circle.

15. PROPOSITION II. *The angle called a radian is of invariable magnitude.*

Let O be the centre of the circle ABC , and let AB be an arc equal in length to the radius OA ; then

$\hat{AOB}$ is a **radian**.

From Euclid VI. 33, we learn that angles at the centre of a circle bear to one another the same ratio as the arcs upon which they stand. Hence

$$\frac{\hat{AOB}}{2 \text{ rt. angles}} = \frac{\text{arc } AB}{\text{semi-circumference}} = \frac{r}{\pi r} = \frac{1}{\pi};$$

$$\therefore \hat{AOB} = \frac{1}{\pi} \times 2 \text{ right angles,}$$

i.e., $1 \text{ radian} = \frac{180}{\pi} \text{ degrees.}$

Hence, π being a constant, a radian is always of the same magnitude, whatever the radius r may be.

16. PROPOSITION III. *To find the number of degrees in a radian.*

$$1 \text{ radian} = \frac{180^\circ}{\pi} = \frac{180^\circ}{3.14159} = 57^\circ.29582 = 57^\circ 17' 45'' \text{ nearly.}$$

17. The relation between a degree and a radian must be very carefully remembered:—

$$180 \text{ degrees} = \pi \text{ radians.}$$

The following equivalents are deduced from this, and must be committed to memory:—

$$90^\circ = \frac{\pi}{2} \text{ radians, } 45^\circ = \frac{\pi}{4} \text{ radians, } 30^\circ = \frac{\pi}{6} \text{ radians, } 60^\circ = \frac{\pi}{3} \text{ radians;}$$

hence the circular measures of 180° , 90° , 45° , 30° , and 60°

are respectively π , $\frac{\pi}{2}$, $\frac{\pi}{4}$, $\frac{\pi}{6}$, and $\frac{\pi}{3}$.

$$\text{Generally, } x^\circ = \frac{\pi x}{180} \text{ radians, and the circ. measure of } x^\circ = \frac{\pi x}{180}.$$

18. When measured in radians, an angle is commonly denoted by a letter of the Greek alphabet such as α , β , γ , θ , ϕ , etc.; and when such a symbol as θ is used to denote an angle, it means

that the angle contains θ radians. Under similar circumstances π denotes an angle containing $3.14159\dots$ radians.

19. If D be the number of degrees, G the number of grades, and θ the number of radians in any angle, it will be seen that θ is connected with D and G by the relation

$$\frac{D}{180} = \frac{G}{200} = \frac{\theta}{\pi},$$

for each of these fractions expresses the ratio which the angle bears to two right angles.

20. To reduce sexagesimal measure to circular we first express the angle in degrees, and we then multiply the number of degrees by $\frac{\pi}{180}$.

$$\begin{aligned} \text{Example. No. of radians in } 23^\circ 15' 45'' &= 23 \frac{15\frac{3}{4}}{60} \times \frac{\pi}{180} \\ &= 23\frac{21}{80} \times \frac{\pi}{180} \\ &= \frac{1861}{14400} \pi. \end{aligned}$$

Similarly to reduce centesimal measure to circular, we express the angle in grades, and we then multiply the number of grades by $\frac{\pi}{200}$.

$$\begin{aligned} \text{Example. No. of radians in } 34^{\text{gr}} 4' 5'' &= 34.0405 \times \frac{\pi}{200} \\ &= .1702025\pi \\ &= \frac{68081\pi}{400000}. \end{aligned}$$

EXAMPLES 3.

Express the following angles in radians in terms of π :—

(1) $\frac{1}{5}$ of a rt. angle.	(6) 10° .	(11) 60° .	(16) $7^\circ 30'$.
(2) $\frac{1}{10}$ of a rt. angle.	(7) 15° .	(12) 135° .	(17) $22^\circ 30'$.
(3) $\frac{1}{3}$ of a rt. angle.	(8) 9° .	(13) 20^{gr} .	(18) $3^\circ 45'$.
(4) $\frac{1}{7}$ of a rt. angle.	(9) 30° .	(14) 25^{gr} .	(19) $6^{\text{gr}} 25'$.
(5) $\frac{1}{6}$ of a rt. angle.	(10) 45° .	(15) 30^{gr} .	(20) $73^{\text{gr}} 75'$.

21. Conversely, to reduce the measure of an angle from radians to degrees or grades, we multiply the number of radians by $\frac{180}{\pi}$ to reduce to *degrees*, and by $\frac{200}{\pi}$ to reduce to *grades*.

$$\begin{array}{lcl}
 \text{Example. } 12 \text{ radians} & = 12 \times \frac{180^\circ}{\pi} & = 12 \times \frac{200^{\text{gr}}}{\pi} \\
 & = 12 \times \frac{180^\circ}{3.1416} & = 12 \times \frac{200^{\text{gr}}}{3.1416} \\
 & = 687.54984^\circ & = 763.9419^{\text{gr}} \\
 & = 687^\circ 32' 59'' \text{ nearly} & = 763^{\text{gr}} 94' 19''.
 \end{array}$$

EXAMPLES 4.

[Take $\pi = 3.1416$, and $\frac{1}{\pi} = .3183$.]

Express in degrees, minutes and seconds, neglecting fractions of a second, the angles whose circular measures are

(1) $\frac{\pi}{2}$.	(6) $\frac{\pi}{5}$.	(11) $\frac{3\pi}{5}$.	(16) $\frac{1}{2}$.
(2) $\frac{\pi}{3}$.	(7) $\frac{\pi}{12}$.	(12) $\frac{2\pi}{7}$.	(17) $\frac{2}{3}$.
(3) $\frac{\pi}{4}$.	(8) $\frac{2\pi}{3}$.	(13) $2\pi^2$.	(18) $.8$.
(4) $\frac{\pi}{6}$.	(9) $\frac{3\pi}{4}$.	(14) $.31416$.	(19) $.3$.
(5) $\frac{\pi}{8}$.	(10) $\frac{5\pi}{6}$.	(15) 5 .	(20) $2\frac{3}{4}$.

Express in centesimal measure an angle of

- (21) $\frac{2}{3}\pi$ radians. (22) $.01$ of a radian. (23) $3\frac{1}{2}$ radians.

22. In Art. 11 the circular measure of an angle was defined to be the *number of radians* in the angle. Thus if $\hat{AOB}$ be one radian, the circular measure of $\hat{AOP}$

$$= \frac{\hat{AOP}}{\hat{AOB}} = \frac{\text{arc AP}}{\text{arc AB}} = \frac{\text{arc subtended}}{\text{radius}}.$$

Hence:—If a circle be described with the angular point as centre, and with any radius, the circular measure of the angle is the ratio which the arc subtending the angle bears to the radius of the circle.

23. If r be the radius of the circle, l the length of the arc which subtends the angle, and θ the circular measure of the angle, we have

$$\theta = \frac{l}{r}, \therefore l = r\theta.$$

Hence, the length of an arc of a circle is obtained by multiplying the number of units of length in the radius by the number of radians in the angle at the centre.

Example. If the radius of a circle be 100 yards, determine the number of yards in an arc which subtends at the centre of the circle an angle of $25^\circ 30'$. ($\pi = 3\frac{1}{2}$.)

$$\frac{\text{Arc}}{\text{radius}} = \text{circular measure of } 25^\circ 30' = 25\frac{1}{2} \times \frac{\pi}{180};$$

$$\therefore \text{length of arc} = 25\frac{1}{2} \times \frac{\pi}{180} \times 100 \text{ yds} = \frac{85\pi}{6} \text{ yds} = 44\frac{11}{12} \text{ yds.}$$

EXAMPLES 5.

[Take $\pi = 2\frac{2}{7}$.]

(1) Find the radius of a circle if an arc a feet long subtends at the centre an angle whose circular measure is m .

(2) Find the radius of a circle when an arc of 1 inch subtends an angle of 1° at the circumference.

(3) The length of the radius of a circle is 6 ft. 3 in. Find approximately the length of an arc which is $\frac{1}{18}$ of the whole circumference.

(4) Find the length of an arc which subtends an angle of $132^\circ 30'$ at the centre of a circle whose radius is 24 feet.

(5) If the radius of a circle is 4000 miles, determine the number of miles in an arc which subtends an angle of $\frac{\pi}{6}$ radians at the centre.

(6) Find the radius of a circle if an arc of 8 ft. 5 in. subtends at the centre an angle of $70^\circ 42'$.

Miscellaneous Examples.

24. The set of miscellaneous examples which follows requires only a knowledge of elementary algebra and geometry.

Example i. If the measured angle of $33^{\circ} 27' 48''$ be represented by 6, what is the unit of measurement?

$$\begin{aligned} 6 \text{ units} &= 33^{\circ} 27' 48'' ; \\ \therefore 1 \text{ unit} &= \frac{1}{6} \text{ of } 33^{\circ} 27' 48'' \\ &= 5^{\circ} 34' 38''. \end{aligned}$$

Example ii. Find the magnitude of an angle of a regular figure of 7 sides.

By Euclid I. 32 (Corollary),

The sum of the 7 interior angles + 4 rt. angles = 14 rt. angles ;

$\therefore$ the sum of the 7 interior angles = 10 rt. angles ;

$\therefore$ each interior angle = $1\frac{2}{7}$ rt. angles
= $128\frac{1}{7}$ degrees.

Example iii. If AC, one side of a right-angled triangle, be 3 inches in length, and the hypotenuse AB be 8 inches, find the length of BC.

By Euclid I. 47 we know that

$$\begin{aligned} AB^2 &= AC^2 + BC^2, \\ \therefore BC^2 &= AB^2 - AC^2, \\ \therefore BC &= \sqrt{64 - 9} \text{ inches} \\ &= \sqrt{55} \text{ inches} \\ &= 7.416 \text{ inches.} \end{aligned}$$

Example iv. If the diameter of a circle be 10 inches, find the distance from the centre of the circle of a chord whose length is 7 inches.

Let AB be the chord, O the centre of the circle, and OC the perpendicular from O upon AB.

Then OB = 5 inches, and BC = $3\frac{1}{2}$ inches ;

$$\begin{aligned} \therefore OC &= \sqrt{OB^2 - BC^2} \\ &= \sqrt{25 - \frac{49}{4}} \text{ inches} \\ &= \frac{1}{2}\sqrt{51} \text{ inches} \\ &= 3.57 \text{ inches nearly.} \end{aligned}$$

EXAMPLES 6. MISCELLANEOUS.

(1) How often is the angle $6^{\circ} 15' 13''$ contained in the angle $62^{\circ} 32' 10''$?

(2) If an angle of $22\frac{1}{2}$ degrees be represented by 5, what angle is the unit of measurement?

- (3) If the measure of $30^{\circ} 30'$ is $\cdot 15$, what is the unit angle?
- (4) Through what angle does an hour-hand revolve in a quarter of an hour?
- (5) The angles of a triangle are in Arithmetical Progression, and one of them is $50^{\circ} 5' 16''$. Find the others.
- (6) Two adjacent angles of a cyclic quadrilateral are $36^{\circ} 17' 25''$ and $125^{\circ} 42' 31''$; what are the other two angles?
- (7) Express in degrees and in radians the sum of the angles of a regular (i) pentagon, (ii) hexagon, (iii) octagon, (iv) decagon.
- (8) Express a single angle of a regular (i) pentagon, (ii) hexagon, (iii) octagon, (iv) decagon, in degrees and in radians.
- (9) Find the number of degrees and the number of radians in the angle between two consecutive spokes of a 14-spoke wheel.
- (10) Find in degrees, and also in grades, the difference between 100° and 100^{gr} .
- (11) An isosceles triangle has each angle at the base double the vertical angle. Express each of the three angles in degrees and in radians.
- (12) The difference between two acute angles of a right-angled triangle is $\frac{2}{5}\pi$ radians: express the angles in radians and also in degrees.
- (13) What angle added to $\cdot 5432$ radians will make up a right angle?
- (14) The angles of a triangle are as $3:4:5$. Express them in degrees and also in radians.
- (15) Express in degrees the angle subtended at the centre of a circle by an arc 15 feet in length, the circumference being 225 feet.
- (16) Find the radius of a circle in which an arc 8 inches long subtends 30° at the centre. $\left(\frac{1}{\pi} = \cdot 3183.\right)$
- (17) The hypotenuse of a right-angled triangle is x inches long and one of the sides is 2 inches; find the length of the other side.
- (18) Determine the length of the shortest side of a right-angled triangle of which the other two sides are 24 and 25 inches.
- (19) In a circle, whose radius is 5 inches, find the distance from the centre of the circle of a chord 6 inches long.

(20) If the radius of a circle be 3 inches, find the length of a side, and the area of a square inscribed in it.

(21) In a circle a chord, $\frac{2}{3}$ of an inch long, is $\frac{1}{3}$ of an inch from the centre; find the diameter of the circle.

(22) The number of grades in one angle is equal to the number of degrees in another. Find the ratio of the magnitudes of the two angles.

(23) The difference between the number of grades and the number of degrees in an angle is 4. Find the angle.

(24) Find two angles such that their difference is one degree, and their sum a radian. ($\pi = \frac{22}{7}$.)

(25) Divide $\frac{2}{3}$ of a right angle into two parts, such that the number of degrees in one part may be to the number of grades in the other part as 3 is to 10.

(26) One angle of a triangle is 45° , and another is $1\frac{1}{2}$ radians. Express the third angle in degrees and also in radians.

(27) The three angles of a triangle are in Arithmetical Progression and the greatest is double the least. Find each angle in degrees.

(28) The three angles of a triangle have the same measure when one is measured in degrees, another in grades and the third in radians; find the third angle in terms of π .

II. THE TRIGONOMETRICAL RATIOS.

Acute Angles.

25. It is proved, in Euclid VI. 4, that two triangles which are equiangular to one another have the sides about equal angles proportional; hence *any* side of the one triangle is in a constant ratio to the *corresponding* side of the other.

We will consider, for example, the triangles POM , $P'OM'$, $P''OM''$, which are formed by taking points P , P' , P'' in either of the lines OA , OB , and from these points drawing the perpendiculars PM , $P'M'$, $P''M''$ upon the other line.

In each of these triangles there is a right angle, viz., at M , M' , M'' , and the angle at O is common to all; hence the triangles are equiangular, and (by Euclid vi. 4) it follows that

$$OM : MP : OP = OM' : M'P' : OP' = OM'' : M''P'' : OP''.$$

From these proportions we see that

$$\frac{MP}{OP} = \frac{M'P'}{OP'} = \frac{M''P''}{OP''}, \quad \frac{OM}{OP} = \frac{OM'}{OP'} = \frac{OM''}{OP''}, \quad \frac{MP}{OM} = \frac{M'P'}{OM'} = \frac{M''P''}{OM''}.$$

Hence, if the angle MOP remains of the same size, then each of the ratios

$$\frac{MP}{OP}, \quad \frac{OM}{OP} \quad \text{and} \quad \frac{MP}{OM}$$

will have an unvarying value, wherever in either bounding line the point P may be taken. These ratios are called respectively the **sine**, **cosine** and **tangent** of the angle AOB .

Their reciprocals, $\frac{OP}{MP}$, $\frac{OP}{OM}$, and $\frac{OM}{MP}$, are respectively called the **cosecant**, **secant** and **cotangent** of the angle AOB .

26. The six ratios mentioned in the preceding article are called the **Trigonometrical Ratios** of the angle AOB . We will now give a formal definition of them.

In an examination, however, the student should draw the four figures on page 52, instead of this one, so that the definitions may apply to an angle of any size.

DEFINITION. In OB , one of the lines which bound the angle AOB , take any point P , and from P draw PM perpendicular to the other bounding line OA , meeting it in M . We thus form a right-angled triangle. Then

$$\text{the sine} \quad \text{of } \hat{AOB} = \frac{MP}{OP} = \frac{\text{perpendicular}}{\text{hypotenuse}};$$

$$\text{the cosine} \quad \text{of } \hat{AOB} = \frac{OM}{OP} = \frac{\text{intercept}}{\text{hypotenuse}};$$

the **tangent** of $\hat{A}OB = \frac{MP}{OM} = \frac{\text{perpendicular}}{\text{intercept}};$

the **cotangent** of $\hat{A}OB = \frac{OM}{MP} = \frac{\text{intercept}}{\text{perpendicular}};$

the **secant** of $\hat{A}OB = \frac{OP}{OM} = \frac{\text{hypotenuse}}{\text{intercept}};$

the **cosecant** of $\hat{A}OB = \frac{OP}{MP} = \frac{\text{hypotenuse}}{\text{perpendicular}}.$

27. The sine, cosine, tangent, cotangent, secant and cosecant of an angle A are written respectively

$\sin A$, $\cos A$, $\tan A$, $\cot A$, $\sec A$, and $\csc A$.

28. DEFINITIONS: (i) The expression $1 - \cos A$ is called the **versed sine** of the angle A , and is written *versin* A .

(ii) The expression $1 - \sin A$ is called the **covered sine** of the angle A , and is written *covers* A .

The Complement of an Angle.

29. DEFINITION. When the sum of two angles is a right angle, each is called the **complement** of the other.

For example,

the complement of $34^\circ 5' 43'' = 90^\circ - 34^\circ 5' 43'' = 55^\circ 54' 17''$,

and the complement of $\frac{\pi}{3} = \frac{\pi}{2} - \frac{\pi}{3} = \frac{3\pi - 2\pi}{6} = \frac{\pi}{6}.$

EXAMPLES 7.

Write down the complements of the following angles:

(1) 10° .	(7) $22^\circ 30'$.	(13) $19^\circ 28' 53''$.	(19) $\frac{\pi}{3}$.
(2) 20° .	(8) $7^\circ 15'$.	(14) $68^\circ 33' 48''$.	(20) $\frac{\pi}{6}$.
(3) 53° .	(9) $43^\circ 5'$.	(15) $15^{\text{gr}} 39'$.	(21) $\frac{\pi}{8}$.
(4) 48° .	(10) $67^\circ 50'$.	(16) $68^{\text{gr}} 54'$.	(22) $\frac{\pi}{4}$.
(5) 39° .	(11) $83^\circ 17'$.	(17) $84^{\text{gr}} 63' 24''$.	(23) $\frac{\pi}{5}$.
(6) 72° .	(12) $25^\circ 14' 38''$.	(18) $37^{\text{gr}} 59' 85''$.	(24) $\frac{2\pi}{5}$.

30. The student must notice in the figure of Art. 26 that each of the angles MOP and MPO is the complement of the other. Also that

- (i) $\frac{\text{MP}}{\text{OP}}$, which is the *sine* of MOP , is also the *cosine* of MPO ,
 i.e., the *sine* of an angle = the *cosine* of its complement;
- (ii) $\frac{\text{OM}}{\text{OP}}$, which is the *cosine* of MOP , is also the *sine* of MPO ,
 i.e., the *cosine* of an angle = the *sine* of its complement;
- (iii) $\frac{\text{MP}}{\text{OM}}$, which is the *tangent* of MOP , is also the *cotangent* of MPO ,
 i.e., the *tangent* of an angle = the *cotan.* of its complement;
- (iv) $\frac{\text{OM}}{\text{MP}}$, which is the *cotangent* of MOP , is also the *tangent* of MPO ,
 i.e., the *cotan.* of an angle = the *tangent* of its complement;
- (v) $\frac{\text{OP}}{\text{OM}}$, which is the *secant* of MOP , is also the *cosecant* of MPO ,
 i.e., the *secant* of an angle = the *cosec.* of its complement;
- (vi) $\frac{\text{OP}}{\text{MP}}$, which is the *cosecant* of MOP , is also the *secant* of MPO ,
 i.e., the *cosec.* of an angle = the *secant* of its complement.

Briefly,

any ratio of an angle = the co-ratio of its complement.

EXAMPLES 8.

Of what acute angle is

(1) $\sin 35^\circ$ the cosine,	(6) $\text{cosec } \frac{\pi}{3}$ the secant,
(2) $\cos 79^\circ$ the sine,	(7) $\tan \frac{\pi}{4}$ the cotangent,
(3) $\tan 43^\circ 28'$ the cotangent,	(8) $\cos 54^\circ 47' 32''$ the sine,
(4) $\cot 87^\circ 41'$ the tangent,	(9) $\tan 86^\circ 15' 9''$ the cotangent,
(5) $\sec \frac{\pi}{6}$ the cosecant,	(10) $\sec 68^\circ 5' 93''$ the cosecant?

The Trigonometrical Ratios of 45° , 60° and 30° .

31. For any given angle each of the six trigonometrical ratios has a definite numerical value. We will determine these

values: (i) for an angle of 45° , (ii) for an angle of 60° , and for an angle of 30° .

(i) **An angle of 45° .**

Let ACB be an isosceles right-angled triangle, the angle ACB being a right angle. Each of the angles ABC and CAB is then half a right angle, i.e., each contains 45 degrees.

Let $AC = x$; then $CB = x$,
and $AB = \sqrt{x^2 + x^2} = x\sqrt{2}$.

Hence

$$\sin 45^\circ = \frac{CB}{AB} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\cos 45^\circ = \frac{AC}{AB} = \frac{x}{x\sqrt{2}} = \frac{1}{\sqrt{2}};$$

$$\tan 45^\circ = \frac{CB}{AC} = \frac{x}{x} = 1;$$

$$\operatorname{cosec} 45^\circ = \frac{AB}{CB} = \frac{x\sqrt{2}}{x} = \sqrt{2};$$

$$\sec 45^\circ = \frac{AB}{AC} = \frac{x\sqrt{2}}{x} = \sqrt{2};$$

$$\cot 45^\circ = \frac{AC}{CB} = \frac{x}{x} = 1.$$

(ii) **An angle of 60° , and an angle of 30° .**

Let ABC be an equilateral triangle, and let AD be drawn perpendicular to BC ; then from Euclid I. 26 it follows that AD bisects BC and bisects also the angle BAC .

Hence $\hat{ABD} = 60^\circ$, and $\hat{BAD} = 30^\circ$.

Let $BD = x$; then $BA = 2x$,
and $DA = \sqrt{4x^2 - x^2} = x\sqrt{3}$.

Hence

$$\sin 60^\circ = \frac{DA}{BA} = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2};$$

$$\cos 60^\circ = \frac{BD}{BA} = \frac{x}{2x} = \frac{1}{2};$$

$$\tan 60^\circ = \frac{DA}{BD} = \frac{x\sqrt{3}}{x} = \sqrt{3};$$

$$\cot 60^\circ = \frac{BD}{DA} = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}};$$

$$\sec 60^\circ = \frac{BA}{BD} = \frac{2x}{x} = 2;$$

$$\operatorname{cosec} 60^\circ = \frac{BA}{DA} = \frac{2x}{x\sqrt{3}} = \frac{2}{\sqrt{3}}.$$

$$\sin 30^\circ = \frac{DB}{AB} = \frac{x}{2x} = \frac{1}{2};$$

$$\cos 30^\circ = \frac{AD}{AB} = \frac{x\sqrt{3}}{2x} = \frac{\sqrt{3}}{2};$$

$$\tan 30^\circ = \frac{DB}{AD} = \frac{x}{x\sqrt{3}} = \frac{1}{\sqrt{3}};$$

$$\cot 30^\circ = \frac{AD}{DB} = \frac{x\sqrt{3}}{x} = \sqrt{3};$$

$$\sec 30^\circ = \frac{AB}{AD} = \frac{2x}{x\sqrt{3}} = \frac{2}{\sqrt{3}};$$

$$\operatorname{cosec} 30^\circ = \frac{AB}{DB} = \frac{2x}{x} = 2.$$

32. All these numerical values the student must know well, so as to be able to recall any one of them instantaneously.

(i) To recall the numerical value of any ratio belonging to an angle of 45° , he should make a **mental picture** of the figure which is here placed in the margin.

The hypotenuse of this right-angled triangle is, as we found in Art. 31, proportional to $\sqrt{2}$, and the sides to 1 and 1; and we may therefore consider them to represent $\sqrt{2}$ units, 1 unit, and 1 unit respectively.

(ii) The trigonometrical ratios of an angle of 60° , and of an angle of 30° , should be remembered in a similar manner.

For both these cases a *mental picture* of the figure in the margin is sufficient. The hypotenuse of this triangle we may consider to represent 2 units, and the sides 1 unit and $\sqrt{3}$ units respectively. In these mental pictures lies a safe and certain method of recalling any or all of the ratios correctly and quickly.

33. We will briefly state these values again:—

	45°	60°	30°		45°	60°	30°
sine	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	cosecant	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	2
cosine	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	secant	$\sqrt{2}$	2	$\frac{2}{\sqrt{3}}$
tangent	1	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	cotangent	1	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$

The following properties will be noticed:

(i) In each case the cotangent, secant and cosecant are respectively the reciprocals of the tangent, cosine and sine.

This also follows necessarily from the definitions given in Art. 26.

(ii) The sine, cosine, and tangent of 60° are respectively equal to the cosine, sine and cotangent of 30° .

This again necessarily follows (Art. 30) from the fact that an angle of 60° and an angle of 30° make up a right angle.

EXAMPLES 9.

Find the value of

(1) $\sin^2 45^\circ + \cos^2 45^\circ$.	(7) $\sin^2 30^\circ - \cos^2 60^\circ$.
(2) $\tan^2 30^\circ + \tan^2 60^\circ$.	(8) $\cot^2 45^\circ + 1$.
(3) $\tan 30^\circ + \cot 30^\circ$.	(9) $\tan 30^\circ \tan 60^\circ \tan 45^\circ$.
(4) $(\sec 60^\circ - 1)(\sec 60^\circ + 1)$.	(10) $\sin 30^\circ \cos 60^\circ + \cos 30^\circ \sin 60^\circ$.
(5) $(\operatorname{cosec} 45^\circ + 1)(\sec 45^\circ - 1)$.	(11) $\cos 45^\circ \sin 45^\circ - \sin^2 30^\circ$.
(6) $1 + \tan^2 30^\circ$.	(12) $(\tan 60^\circ - \tan 30^\circ) \div (1 + \tan 60^\circ \tan 30^\circ)$.

The Trigonometrical Ratios of 0° and 90° .34. (i) An angle of 0° .

For an angle of 0° the rotating line coincides with the initial line OX . We will however imagine it to occupy a neighbouring position such as OA , and we will ascertain what happens as the angle XOA diminishes.

It will be seen that as the angle XOA diminishes, (OP remaining of constant length,) the length OM *increases*, and the length MP *diminishes*.

Ultimately, when OA falls upon OX , OP and OM coincide, and MP vanishes altogether.

$$\begin{array}{lcl}
 \text{Hence } \sin 0^\circ = \frac{MP}{OP} = \frac{0}{OP} = 0; & & \operatorname{cosec} 0^\circ = \frac{OP}{MP} = \frac{OP}{0} = \infty; \\
 \cos 0^\circ = \frac{OM}{OP} = \frac{OP}{OP} = 1; & & \sec 0^\circ = \frac{OP}{OM} = \frac{OP}{OP} = 1; \\
 \tan 0^\circ = \frac{MP}{OM} = \frac{0}{OP} = 0; & & \cot 0^\circ = \frac{OM}{MP} = \frac{OP}{0} = \infty.
 \end{array}$$

NOTE.—If the denominator of a fractional quantity diminishes, while the numerator remains constant, the magnitude of the quantity increases; and if the denominator becomes indefinitely small, the quantity becomes indefinitely great.

The numerical value of a quantity which is indefinitely great is called **infinity**, and is represented by the symbol ∞ .

(ii) An angle of 90° .

For an angle of 90° the rotating line coincides with OY : but we will again suppose it to occupy a neighbouring position, such as OA , and we will ascertain what happens as it approaches OY .

It will be seen that as OA moves nearer to OY , (OP remaining constant in length,) the length MP increases, and the length OM diminishes.

Ultimately, when OA falls upon OY , OP and MP coincide, and OM vanishes altogether.

$$\begin{array}{l|l} \text{Hence } \sin 90^\circ = \frac{MP}{OP} = \frac{OP}{OP} = 1; & \operatorname{cosec} 90^\circ = \frac{OP}{MP} = \frac{OP}{OP} = 1; \\ \cos 90^\circ = \frac{OM}{OP} = \frac{0}{OP} = 0; & \sec 90^\circ = \frac{OP}{OM} = \frac{OP}{0} = \infty; \\ \tan 90^\circ = \frac{MP}{OM} = \frac{OP}{0} = \infty; & \cot 90^\circ = \frac{OM}{MP} = \frac{0}{OP} = 0. \end{array}$$

35. The values of the trigonometrical ratios of 0° and 90° are of great importance, and must be remembered; but the student would do well not to attempt to commit them to memory as isolated or disconnected facts. As in the case of 45° , 60° , and 30° , he should call up a **mental picture** of what happens; how the intercept vanishes in the case of 90° , and the perpendicular in the case of 0° ; and how in each case the lengths which do not vanish become equal.

General Remarks.

36. If one angle of a triangle is a right angle it is greater than either of the other two angles; consequently, the side opposite to a right angle is greater than either of the other two sides. Hence, the *sine* or the *cosine* of an angle can in no case be numerically greater than 1; and in no case can a *secant* or a *cosecant* be numerically less than 1.

On the other hand, there is no restriction as regards the ratio which one of the sides containing the right angle may bear to the other; consequently, there are no limits to the numerical value of a *tangent* or a *cotangent*: it may be infinitely great, as in

the case of $\tan 90^\circ$, or indefinitely small, as in the case of $\tan 0^\circ$; and, for angles intermediate between 90° and 0° , it may have any value intermediate between ∞ and 0.

37. Turning again to the figure in Art. 34, it will be seen that as the angle AOB increases, the line MP will *increase*; hence, since OP remains unchanged, the ratio of MP to OP will increase as the angle increases from 0° to 90° . On the other hand, as the angle AOB increases, the line OM will *decrease*; hence, the ratio of OM to OP will *decrease* as the angle increases from 0° to 90° .

Consequently, of any two angles between 0° and 90° the greater angle has a *greater* sine, but a *smaller* cosine.

38. It will be understood that the value of any Trigonometrical Ratio can be determined for any angle whatever. They are all recorded in books of "Mathematical Tables." Only those which we specially determine need be committed to memory.

39. A power of a trigonometrical ratio is denoted in the same way as a power of a number, but the index is placed between the name of the ratio and the angle: for example, the 5th power of $\sin A$ is written thus:— $\sin^5 A$.

40. The following examples will help the student to become familiar with the numerical values which we have determined.

Example i. Find the value of

$$(\sin 90^\circ - \cos 90^\circ + \sec 60^\circ) (\cos 90^\circ + \sec 45^\circ + \tan 30^\circ).$$

$$\begin{aligned} \text{Value} &= (1 - 0 + 2) \left(0 + \sqrt{2} + \frac{1}{\sqrt{3}} \right) \\ &= 3 \times \left(\sqrt{2} + \frac{1}{\sqrt{3}} \right) \\ &= 3\sqrt{2} + \sqrt{3}. \end{aligned}$$

Example ii. Prove that $4 \cos^3 \frac{\pi}{6} - 3 \cos \frac{\pi}{6} = 3 \cos \frac{\pi}{3} - 4 \cos^3 \frac{\pi}{3} - 1$.

When neither side of the equality is in a simple form, we may simplify them separately and in turn as below:—

$$4 \cos^3 \frac{\pi}{6} - 3 \cos \frac{\pi}{6} = 4 \times \left(\frac{\sqrt{3}}{2} \right)^3 - 3 \times \frac{\sqrt{3}}{2} = 4 \times \frac{3}{4} \times \frac{\sqrt{3}}{2} - \frac{3\sqrt{3}}{2} = 0;$$

$$\text{and } 3 \cos \frac{\pi}{3} - 4 \cos^3 \frac{\pi}{3} - 1 = 3 \times \frac{1}{2} - 4 \times \left(\frac{1}{2} \right)^3 - 1 = \frac{3}{2} - \frac{1}{2} - 1 = 0;$$

$$\therefore 4 \cos^3 \frac{\pi}{6} - 3 \cos \frac{\pi}{6} = 3 \cos \frac{\pi}{3} - 4 \cos^3 \frac{\pi}{3} - 1.$$

EXAMPLES 10.

Find the numerical value of

- (1) $\cos^2 30^\circ - \cos^2 45^\circ + \cos^2 0^\circ$. (2) $\sin 45^\circ \times \cos 60^\circ \times \cos 0^\circ$.
 (3) $\tan 45^\circ \times \tan 0^\circ \div \tan 60^\circ$.
 (4) $(\sin 90^\circ - \sin 45^\circ)(\sin 90^\circ + \sin 45^\circ)$.
 (5) $\tan^2 60^\circ - 2 \tan^2 45^\circ$. (6) $\cot^2 60^\circ + \sin^2 30^\circ - \cos^2 90^\circ$.
 (7) $\cot^2 30^\circ - 2 \sin^2 30^\circ - \frac{3}{4} \operatorname{cosec}^2 45^\circ$.
 (8) $\cot \frac{\pi}{6} \times \cot \frac{\pi}{4} \times \cot \frac{\pi}{3}$. (9) $\sin \frac{\pi}{3} \cos \frac{\pi}{6} + \cos \frac{\pi}{3} \sin \frac{\pi}{6}$.
 (10) $\sin^2 \frac{\pi}{3} + \cos^2 \frac{\pi}{2} + \tan^2 \frac{\pi}{4}$. (11) $\sin^2 \frac{\pi}{4} + \cos^2 \frac{\pi}{3} - \frac{1}{2} \sin \frac{\pi}{2}$.
 (12) $4 \cos^2 \frac{\pi}{3} - \cos^2 \frac{\pi}{4} - \cos^2 \frac{\pi}{4} \times \cos^2 \frac{\pi}{3}$.

Prove that

- (13) $3 \tan 60^\circ - \cot^3 30^\circ = 0$.
 (14) $3 \cot^2 60^\circ - \sin^2 45^\circ = \cos 60^\circ$.
 (15) $\frac{\sin 30^\circ}{\cot 45^\circ} + \frac{\tan 45^\circ}{\operatorname{cosec} 30^\circ} = \sin^2 60^\circ + \sin^2 30^\circ$.
 (16) $\cos^2 \frac{\pi}{3} : \cos^2 \frac{\pi}{4} : \cos^2 \frac{\pi}{6} = 1 : 2 : 3$.

Geometrical Problems.

41. We will now use the values of the trigonometrical ratios which have been determined, in the solution of a few simple geometrical problems.

The angle $\angle BCA$ in the figure of Example i. is called the **angle of elevation** of the point A when viewed from C , for it is the angle through which a horizontal telescope at C must be turned upwards in order to bring A into the field of view. This angle is equal to $\angle DAC$, the **angle of depression** of C when viewed from A .

The angle $\angle BCA$ may be called also the *angular altitude* of A , and also the *angle subtended* at C by the tower.

Angles of elevation, and angles in a horizontal plane are measured by an instrument called a **Theodolite**. Other angles subtended at the eye of an observer are measured by a **Sextant**.

Example i. At a place C, on the ground, the angle of elevation of the top A of a tower AB is observed to be 60° . The distance between C and the foot of the tower is 30 yards. Find the height of the tower.

Let h = no. of yds in the height BA, then

$$\frac{h}{30} = \tan 60^\circ;$$

$$\therefore \frac{h}{30} = \sqrt{3};$$

$$\therefore h = 30\sqrt{3};$$

$$\therefore \text{height of tower} = 30\sqrt{3} \text{ yards} = 51.96 \text{ yards.}$$

Example ii. From a point A on the ground on one side of a street, the height of a house BD on the opposite side is observed to subtend an angle of 60° , and the top C of a window is at an angular altitude of 45° . If the street is 40 feet wide, determine the height of the house and the distance of the top of the window from the ground.

Let $BC = x$ feet and $CD = y$ feet.

$$\text{Then } \frac{x}{40} = \tan 45^\circ = 1, \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$$\text{and } \frac{x+y}{40} = \tan 60^\circ = \sqrt{3}; \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$$\therefore x = 40, \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$$\text{and } x + y = 40\sqrt{3}; \quad \left. \begin{array}{l} \\ \end{array} \right\}$$

$\therefore$ the height of the house $= 40\sqrt{3}$ feet $= 69.28$ feet,
and the height of the top of the window $= 40$ feet.

NOTE. In problems of this kind, which involve both algebra and geometry, the lengths of lines should be expressed by algebraic symbols, when they are unknown and have to be determined. It is then necessary merely to write down a number of equations sufficient to determine the unknown quantities, and the remainder of the process is purely algebraical. Also the numerical values of the known lengths or angles, and the symbols for the unknown, should be distinctly marked in the figure.

EXAMPLES 11.

[Given $\sqrt{2} = 1.41421\dots$; $\sqrt{3} = 1.73205\dots$]

(1) How far must a man walk away from a house 28 feet high in order that it may subtend an angle of 45° ?

(2) What is the height of a house whose angular altitude, at a point in the street 35 feet away, is 60° ?

(3) A church tower subtends an angle of 30° at a point on the horizontal plane through its base. If the height of the tower be 150 feet, find its horizontal distance from the observer.

(4) How far from a pillar, 55 feet high, must an observer stand that it may have an angular elevation of 60° ?

(5) The upper part of a tree broken over by the wind makes an angle of 30° with the ground, and the distance from the root to the top of the stump is 50 feet: what was the height of the tree?

(6) A tower casts a shadow 240 feet long when the sun is at an altitude of 30° . Find the height of the tower.

(7) A pillar 40 feet high casts a shadow 30 feet long, find the cosine of the sun's angular altitude.

(8) From the top of a tower 215 feet high the angle of depression of an object on the ground is 30° ; how far is the object from the foot of the tower, and how far from the observer?

(9) A lark is seen by one observer to be directly over his head. At the same time another observer, 40 yards away, calculates its angular altitude to be 60° . Find the height of the lark above the ground, and its distance from the second observer.

(10) The angular altitude of a tower at a place 200 yards from its base being 30° , what is the height of the tower, and the distance of its top from the place of observation?

(11) An observer is directly opposite a post on the other side of a stream. He walks 100 feet along the bank and finds himself directly opposite another post. If the line between the two posts subtends an angle of 30° at either place of observation, determine the width of the stream.

(12) The look-out at a ship's mast-head, 90 feet from the surface of the water, catches sight of an object, the tangent of whose angle of depression is $\frac{1}{5}$. Find the distance of the object from the ship and also from the mast-head. ($\sqrt{26} = 5.099\dots$)

(13) There are two trees on a horizontal plane. If I stand at the foot of the first the angle of elevation of the top of the other is 60° , but when I stand at the foot of the second the angle of elevation of the top of the first is 30° . Prove that the second tree is three times as high as the first.

(14) A man observes at two points the angles of elevation of a tower which he knows to be 100 feet high, and finds them to be 60° and 30° : determine the distance between the points of observation, supposing them to be in a line with the foot of the tower.

(15) At a certain distance from the foot of a tower its elevation is observed to be 45° , and at a place 30 yards nearer it is 60° . Find the height of the tower, and the distance of its top from the second place of observation.

(16) A tower has an elevation of 45° at a point on the horizontal plane. After walking 173 feet farther away from the tower, the observer finds the elevation to be 30° . Find the height of the tower, and the distance of its top from the first place of observation.

(17) A steeple-jack, ascending a church spire, is seen at a place 150 yards away to be at an angular altitude of 30° . Half-an-hour later his angular altitude is 45° ; what distance has he climbed in the interval?

(18) A man in a balloon vertically over a straight horizontal road observes that the angles of depression of two consecutive milestones on the road are 45° and 30° . Find the height of the balloon above the road.

Miscellaneous Propositions.

42. *To express any one of the trigonometrical ratios in terms of any other.*

The method will be sufficiently clear if we express each ratio in terms of (i) the sine, (ii) the tangent. Moreover we need only consider the sine, cosine and tangent, since the cosecant, secant and cotangent are respectively the reciprocals of these; and we will suppose the angle to be acute.

(i) Let $\angle A = A$.

Then $\sin A = \frac{MP}{OP}$.

We take the denominator OP as the unit of length; it follows that

$$\frac{MP}{1} = \sin A,$$

$$\therefore MP = \sin A \text{ units,}$$

and $\therefore OM = \sqrt{OP^2 - MP^2} = \sqrt{1 - \sin^2 A}$ units.

We mark in the figure the number of units in each line respectively. Hence we see that

$$\cos A = \frac{OM}{OP} = \sqrt{1 - \sin^2 A}, \text{ and } \tan A = \frac{MP}{OM} = \frac{\sin A}{\sqrt{1 - \sin^2 A}}.$$

(ii) Let $\hat{XOA} = A$, then $\tan A = \frac{MP}{OM}$.

We take the denominator OM as the unit of length; it follows that

$$\frac{MP}{1} = \tan A,$$

$$\therefore MP = \tan A \text{ units,}$$

and $\therefore OP = \sqrt{OM^2 + MP^2}$
 $= \sqrt{1 + \tan^2 A} \text{ units.}$

We mark in the figure the number of units in each line respectively. Hence we see that

$$\sin A = \frac{MP}{OP} = \frac{\tan A}{\sqrt{1 + \tan^2 A}}, \quad \cos A = \frac{OM}{OP} = \frac{1}{\sqrt{1 + \tan^2 A}}.$$

43. When the numerical value of one trigonometrical ratio of an angle is given, we may by the method of Art. 42 determine the numerical values of all the other ratios.

Example i. Given $\cos A = \frac{3}{4}$, find the numerical values of the other trigonometrical ratios of A .

Let $\hat{XOA} = A$; and, since $\cos A = \frac{3}{4}$, let us choose for unit of length $\frac{1}{4}$ of the line OP .

Then $OP = 4$ units, and $OM = 3$ units;

$$\therefore MP = \sqrt{16 - 9} \text{ units} = \sqrt{7} \text{ units.}$$

Marking these measures in the figure, we see that

$$\sin A = \frac{\sqrt{7}}{4}; \quad \tan A = \frac{\sqrt{7}}{3}; \quad \cot A = \frac{3}{\sqrt{7}};$$

$$\sec A = \frac{4}{3}; \quad \operatorname{cosec} A = \frac{4}{\sqrt{7}}.$$

EXAMPLES 12.

Express the trigonometrical ratios of an acute angle A in terms of

(1) the cosine.

(3) the secant.

(2) the cotangent.

(4) the cosecant.

Find the numerical values of the other ratios when

$$(5) \sin A = \frac{1}{3}. \quad (8) \tan A = \frac{3}{8}. \quad (11) \sin A = \frac{\sqrt{2}}{5}.$$

$$(6) \sin A = \frac{m}{n}. \quad (9) \operatorname{cosec} A = \frac{3}{2}. \quad (12) \cot A = \sqrt{5}.$$

$$(7) \tan A = 7. \quad (10) \cos A = \frac{m}{n}. \quad (13) \operatorname{cosec} A = \sqrt{3}.$$

44. To construct an acute angle having (i) a given tangent, (ii) a given cosine, (iii) a given sine.

(i) **An angle which has a given tangent.**

Let $\frac{a}{b}$ be the given tangent.

Take a straight line OA to represent b units of length; and at A draw AB perpendicular to OA , and equal in length to a units. Join OB .

$$\text{Then } \tan AOB = \frac{AB}{OA} = \frac{a}{b};$$

$\therefore AOB$ is the angle required.

(ii) **An angle which has a given cosine.**

Let $\frac{a}{b}$ be the given cosine.

Take the straight line OA to represent a units of length; and with centre O , and with a radius equal in length to b units, describe a circle.

At A draw AB perpendicular to OA , and let it meet the circle at B . Join OB .

$$\text{Then } \cos AOB = \frac{OA}{OB} = \frac{a}{b};$$

$\therefore AOB$ is the angle required.

(iii) An angle which has a given sine.

Let $\frac{a}{b}$ be the given sine.

Take a straight line AB to represent a units of length; and with centre B, and with a radius equal in length to b units, describe a circle.

At A draw AO perpendicular to AB, and let it meet the circle at O. Join OB.

$$\text{Then } \sin \angle AOB = \frac{AB}{OB} = \frac{a}{b};$$

$\therefore \angle AOB$ is the angle required.

NOTE. If the given ratio be the cotangent, secant or cosecant, we proceed as in the case of the tangent, cosine or sine respectively.

45. The given value of a ratio may involve surds such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ &c. In such cases we may determine the lengths of the corresponding lines by repeated applications of Euclid i. 47.

For example, let AB represent one unit of length; and let BC, CD, DE, EF, FG, GH be drawn, each equal to AB, and perpendicular respectively to AB, AC, AD, AE, AF, AG.

It follows from Euclid i. 47, that AB, AC, AD, AE, AF, AG, AH contain respectively 1, $\sqrt{2}$, $\sqrt{3}$, 2, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$ units of length.

EXAMPLES 13.

Construct an acute angle

whose sine is	(1) $\frac{3}{4}$.	(2) $\frac{1}{4}\sqrt{3}$.	(3) $\frac{\sqrt{5}}{\sqrt{7}}$;
whose cosine is	(4) $\frac{2}{3}$.	(5) $\frac{1}{3}\sqrt{5}$.	(6) $\frac{\sqrt{3}}{\sqrt{5}}$;
whose tangent is	(7) $\frac{5}{2}$.	(8) $\frac{1}{2}\sqrt{2}$.	(9) $\frac{\sqrt{7}}{\sqrt{2}}$.

III. IDENTITIES.

46. We will now consider certain relations which exist between two or more ratios of the same angle. We will call the angle A , but the relations will be true for all angles whatever.

(i) To prove $\tan A = \frac{\sin A}{\cos A}$.

With the usual figure,

$$\tan A = \frac{MP}{OM} = \frac{\frac{MP}{OP}}{\frac{OM}{OP}} = \frac{\sin A}{\cos A}.$$

(ii) To prove $\sin^2 A + \cos^2 A = 1$.

With the same figure,

$$\sin^2 A + \cos^2 A = \frac{MP^2}{OP^2} + \frac{OM^2}{OP^2} = \frac{MP^2 + OM^2}{OP^2} = \frac{OP^2}{OP^2} = 1.$$

(iii) To prove $1 + \tan^2 A = \sec^2 A$.

With the same figure,

$$1 + \tan^2 A = 1 + \frac{MP^2}{OM^2} = \frac{OM^2 + MP^2}{OM^2} = \frac{OP^2}{OM^2} = \sec^2 A.$$

(iv) To prove $1 + \cot^2 A = \operatorname{cosec}^2 A$.

With the same figure,

$$1 + \cot^2 A = 1 + \frac{OM^2}{MP^2} = \frac{MP^2 + OM^2}{MP^2} = \frac{OP^2}{MP^2} = \operatorname{cosec}^2 A.$$

NOTE. It must be remembered that

$$\cot A = \frac{1}{\tan A} = \frac{\cos A}{\sin A}, \quad \sec A = \frac{1}{\cos A}, \quad \text{and} \quad \operatorname{cosec} A = \frac{1}{\sin A}.$$

Also that $\sin$, $\cos$, $\tan$, &c. have no meaning when separated from the angle; they must not be treated as multipliers. For example, $\frac{\sin}{\cos} A$ has no meaning, and may not be written for $\tan A$.

47. When one trigonometrical expression can be deduced from another by means of the general formulæ which have already been proved, or will hereafter be proved, the two expressions are said to be identically equal, and the equation which states that the one expression is equal to the other is called an **identical equation**, or briefly an **identity**.

In the following examples the object is to transform the expression on the left-hand side of the equation by means of the formulæ which have been proved; and to do so in such a way that the expression on the right-hand side may result from it. In many cases it will be found best to get rid of the ratios tangent, cotangent, secant, and cosecant by substituting for them their respective equivalents in terms of the ratios *sine* and *cosine*; but it should be understood that there are no definite rules to follow.

Example i. Prove that $\sec^2 A + \operatorname{cosec}^2 A = \sec^2 A \operatorname{cosec}^2 A$.

$$\begin{aligned}\sec^2 A + \operatorname{cosec}^2 A &= \frac{1}{\cos^2 A} + \frac{1}{\sin^2 A} = \frac{\sin^2 A + \cos^2 A}{\sin^2 A \cos^2 A} \\ &= \frac{1}{\sin^2 A \cos^2 A} = \frac{1}{\cos^2 A} \times \frac{1}{\sin^2 A} \\ &= \sec^2 A \operatorname{cosec}^2 A.\end{aligned}$$

Example ii. Prove that $\sin^4 A - \cos^4 A = 1 - 2 \cos^2 A$.

$$\begin{aligned}\sin^4 A - \cos^4 A &= (\sin^2 A + \cos^2 A)(\sin^2 A - \cos^2 A) \\ &= \sin^2 A - \cos^2 A \\ &= 1 - \cos^2 A - \cos^2 A \\ &= 1 - 2 \cos^2 A.\end{aligned}$$

Example iii. Prove that $\tan^2 A + \cot^2 A + 2 = \sec^2 A \operatorname{cosec}^2 A$.

$$\begin{aligned}\tan^2 A + \cot^2 A + 2 &= (\tan^2 A + 1)(\cot^2 A + 1) \\ &= \sec^2 A \operatorname{cosec}^2 A.\end{aligned}$$

In proving an identity the left-hand member must generally be dealt with until the right-hand member results from it; but if the right-hand member is not in a simple form the student may proceed as in Art. 40, Ex. ii.

EXAMPLES 14.

Prove that

- (1) $\sin A \sec A = \tan A$. (2) $\sin A \tan^2 A \cot^3 A = \cos A$.
- (3) $\sin^2 A - \cos^2 B = \sin^2 B - \cos^2 A$.
- (4) $\cot^2 A - \cos^2 A = \cos^2 A \cot^2 A$.
- (5) $\tan^2 A - \sin^2 A = \sin^4 A \sec^2 A$.
- (6) $\tan A + \cot A = \sec A \operatorname{cosec} A$.
- (7) $\sin^4 A - \cos^4 A = \sin^2 A - \cos^2 A$.
- (8) $\cos^2 A + \cos^4 A \operatorname{cosec}^2 A = \cot^2 A$.
- (9) $\sec A \cot^3 A (\sec^2 A - 1) = \operatorname{cosec} A$.
- (10) $\sin^2 A + \sin^2 A \tan^2 A = \tan^2 A$.

$$(11) \quad \sin^2 A + \sin^2 A \cot^2 A = 1.$$

$$(12) \quad (\sin A + \cos A)^2 = 1 + 2 \sin A \cos A.$$

$$(13) \quad (\sin A - \cos A)^2 = 1 - 2 \sin A \cos A.$$

$$(14) \quad \frac{\tan A + \tan B}{\cot A + \cot B} = \tan A \tan B.$$

$$(15) \quad \frac{\cos x}{1 - \sin x} + \frac{\cos x}{1 + \sin x} = 2 \sec x.$$

EXAMPLES 15.

Prove that

$$(1) \quad \cos^2 A \cos^2 B - \sin^2 A \sin^2 B = \cos^2 A - \sin^2 B.$$

$$(2) \quad \tan^2 x - \cot^2 y = \frac{\sin^2 x - \cos^2 y}{\cos^2 x \sin^2 y}.$$

$$(3) \quad 1 + \frac{\tan^2 A}{1 + \sec A} = \sec A. \quad (4) \quad \frac{(1 - \operatorname{versin} A)^2 + \sin^2 A}{1 + \cot^2 A} = \sin^2 A.$$

$$(5) \quad \frac{\tan A (\cot A + 1)}{1 - \tan A} = \frac{\cos A + \sin A}{\cos A - \sin A}.$$

$$(6) \quad \sin^2 A \tan^2 A + \cos^2 A \cot^2 A = \tan^2 A + \cot^2 A - 1.$$

$$(7) \quad \frac{(1 + \cos^2 A)^2 + (1 - \cos^2 A)^2}{(1 + \cos^2 A)^2 - (1 - \cos^2 A)^2} = \frac{1}{2} (\cos^2 A + \sec^2 A).$$

$$(8) \quad \frac{\sec x - \tan x}{\operatorname{cosec} x + \cot x} - \frac{\operatorname{cosec} x - \cot x}{\sec x + \tan x} = 0.$$

$$(9) \quad \frac{\cos x + \cos y}{\sin x + \sin y} + \frac{\sin x - \sin y}{\cos x - \cos y} = 0.$$

$$(10) \quad (\sin A + \cos A) (\tan A + \cot A) = \sec A + \operatorname{cosec} A.$$

$$(11) \quad (1 + \sin A - \cos A)^2 + (1 - \sin A + \cos A)^2 = 4 (1 - \sin A \cos A).$$

$$(12) \quad (1 + \sin A + \cos A)^2 = 2 (1 + \sin A) (1 + \cos A).$$

$$(13) \quad \sin^4 A + \cos^4 A = 1 - 2 \sin^2 A \cos^2 A.$$

$$(14) \quad \tan^2 A + \cot^2 A = \frac{1 - 2 \sin^2 A \cos^2 A}{\sin^2 A - \sin^4 A}.$$

IV. EXPRESSIONS INVOLVING TWO ANGLES.

48. We will next proceed to determine the trigonometrical ratios of the sum and of the difference of two angles in terms of the ratios of the two angles themselves; i.e., we will determine the ratios of the angles $(A + B)$ and $(A - B)$ in terms of those of

A and B. We will still suppose that each of the angles considered, namely, A , B , $(A + B)$, and $(A - B)$, is less than a right angle, but the results are absolutely true, whatever the magnitude of the angles may be.

The Sine, Cosine and Tangent of $(A + B)$.

49. (i) *To prove that $\sin(A + B) = \sin A \cos B + \cos A \sin B$.*

Let $\hat{XOA} = A$
and $\hat{AOB} = B$ } , then $\hat{XOB} = A + B$.

In OB , the line which bounds the angle $(A + B)$, take any point Q ; and from Q draw QP perpendicular to OA .

Also from Q and P respectively draw QN and PM perpendicular to OX ; and from P draw PR perpendicular to QN .

Then $\hat{QRP} = 90^\circ$,
 $\therefore \hat{RQP} = 90^\circ - \hat{QPR} = \hat{RPO} = \hat{XOP} = A$;

$$\begin{aligned} \therefore \sin(A + B) &= \frac{NQ}{OQ} = \frac{MP + RQ}{OQ} \\ &= \frac{MP}{OQ} + \frac{RQ}{OQ} \\ &= \frac{MP}{OP} \cdot \frac{OP}{OQ} + \frac{RQ}{QP} \cdot \frac{QP}{OQ} \\ &= \sin XOA \cos AOB + \cos RQP \sin AOB \\ &= \sin A \cos B + \cos A \sin B. \end{aligned}$$

(ii) *To prove that $\cos(A + B) = \cos A \cos B - \sin A \sin B$.*

Using the same figure, we see that

$$\begin{aligned} \cos(A + B) &= \frac{ON}{OQ} = \frac{OM - RP}{OQ} = \frac{OM}{OQ} - \frac{RP}{OQ} \\ &= \frac{OM}{OP} \cdot \frac{OP}{OQ} - \frac{RP}{QP} \cdot \frac{QP}{OQ} \\ &= \cos XOA \cos AOB - \sin RQP \sin AOB \\ &= \cos A \cos B - \sin A \sin B. \end{aligned}$$

50. To prove that $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$.

$$\begin{aligned}\tan(A+B) &= \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A \cos B + \cos A \sin B}{\cos A \cos B - \sin A \sin B} \\ &= \frac{\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B}}{1 - \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}} = \frac{\tan A + \tan B}{1 - \tan A \tan B}.\end{aligned}$$

NOTE. We may again warn beginners against the mistake of treating the prefixes *sin*, *cos*, &c. as *multipliers*; for example, $\sin(A+B)$ is **not** equivalent to $\sin A + \sin B$.

The Sine, Cosine and Tangent of 75° .

51. As an example of the use of the formulæ of Arts. 49 and 50, we will suppose $A = 45^\circ$, and $B = 30^\circ$; then $A+B = 75^\circ$.

Hence, (i) $\sin 75^\circ = \sin(45^\circ + 30^\circ)$

$$\begin{aligned}&= \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}}.\end{aligned}$$

(ii) $\cos 75^\circ = \cos(45^\circ + 30^\circ)$

$$\begin{aligned}&= \cos 45^\circ \cos 30^\circ - \sin 45^\circ \sin 30^\circ \\ &= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \cdot \frac{1}{2} \\ &= \frac{\sqrt{3}-1}{2\sqrt{2}}.\end{aligned}$$

(iii) $\tan 75^\circ = \tan(45^\circ + 30^\circ)$

$$\begin{aligned}&= \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} \\ &= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \\ &= 2 + \sqrt{3}.\end{aligned}$$

These values must be remembered.

The Sine, Cosine and Tangent of $(A - B)$.

52. (i) To prove that $\sin(A - B) = \sin A \cos B - \cos A \sin B$.

Let $\hat{XOA} = A$
and $\hat{AOB} = B$ } , then $\hat{XOB} = A - B$.

In OB , the line which bounds the angle $(A - B)$, take any point Q ; and from Q draw QP perpendicular to OA .

Also from Q and P respectively draw QN and PM perpendicular to OX ; and from P draw PR perpendicular to NQ produced.

Then $\hat{PRQ} = 90^\circ$,
and $\hat{RQP} = \hat{QPM} = 90^\circ - \hat{OPM} = \hat{MOP} = A$; }

$$\therefore \sin(A - B) = \frac{NQ}{OQ} = \frac{MP - QR}{OQ} = \frac{MP}{OQ} - \frac{QR}{OQ}$$

$$= \frac{MP}{OP} \cdot \frac{OP}{OQ} - \frac{QR}{PQ} \cdot \frac{PQ}{OQ}$$

$$= \sin XOA \cos AOB - \cos RQP \sin AOB$$

$$= \sin A \cos B - \cos A \sin B.$$

(ii) To prove that $\cos(A - B) = \cos A \cos B + \sin A \sin B$.

Using the same figure, we see that

$$\cos(A - B) = \frac{ON}{OQ} = \frac{OM + PR}{OQ} = \frac{OM}{OQ} + \frac{PR}{OQ}$$

$$= \frac{OM}{OP} \cdot \frac{OP}{OQ} + \frac{PR}{PQ} \cdot \frac{PQ}{OQ}$$

$$= \cos XOA \cos AOB + \sin RQP \sin AOB$$

$$= \cos A \cos B + \sin A \sin B.$$

53. To prove that $\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$.

$$\begin{aligned} \text{Thus } \tan(A - B) &= \frac{\sin(A - B)}{\cos(A - B)} = \frac{\sin A \cos B - \cos A \sin B}{\cos A \cos B + \sin A \sin B} \\ &= \frac{\frac{\sin A}{\cos A} - \frac{\sin B}{\cos B}}{1 + \frac{\sin A}{\cos A} \cdot \frac{\sin B}{\cos B}} = \frac{\tan A - \tan B}{1 + \tan A \tan B}. \end{aligned}$$

The Sine, Cosine and Tangent of 15° .

54. To illustrate the use of the expressions obtained for the sine, cosine and tangent of $(A - B)$, we will suppose A to be an angle of 60° , and B an angle of 45° . Then $(A - B)$ is an angle of 15° . It will be found, as in Art. 51, that

$$\sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

$$\cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}.$$

$$\tan 15^\circ = 2 - \sqrt{3}.$$

These values must be remembered.

The Trigonometrical Ratios of 15° and 75° .

55. These values should be compared, and it should be noticed that:

	<i>sine</i>	<i>cosine</i>	<i>tangent</i>	<i>cotangent</i>
15°	$\frac{\sqrt{3} - 1}{2\sqrt{2}}$	$\frac{\sqrt{3} + 1}{2\sqrt{2}}$	$2 - \sqrt{3}$	$2 + \sqrt{3}$
75°	$\frac{\sqrt{3} + 1}{2\sqrt{2}}$	$\frac{\sqrt{3} - 1}{2\sqrt{2}}$	$2 + \sqrt{3}$	$2 - \sqrt{3}$

(i) The sine, cosine and tangent of the one angle are respectively equal to the cosine, sine, and cotangent of the other. This would of course be expected, for 15° and 75° are *complementary* angles (Art. 29);

(ii) The resemblance between $\frac{\sqrt{3}-1}{2\sqrt{2}}$ and $\frac{\sqrt{3}+1}{2\sqrt{2}}$, and between $2-\sqrt{3}$ and $2+\sqrt{3}$, will perhaps be of some help to the student in remembering to which ratio of which angle they respectively belong.

Thus, $\frac{\sqrt{3}+1}{2\sqrt{2}}$ being greater than $\frac{\sqrt{3}-1}{2\sqrt{2}}$, the former must necessarily be the sine of the *greater* angle and the cosine of the *smaller* angle; i.e., $\frac{\sqrt{3}+1}{2\sqrt{2}}$ is the sine of 75° and the cosine of 15° .

On the other hand, $\frac{\sqrt{3}-1}{2\sqrt{2}}$ being less than $\frac{\sqrt{3}+1}{2\sqrt{2}}$, the former must necessarily be the sine of the *smaller* angle and the cosine of the *greater*; i.e. $\frac{\sqrt{3}-1}{2\sqrt{2}}$ is the sine of 15° and the cosine of 75° .

Again $2+\sqrt{3}$, being the *greater*, is the tangent of 75° ; and $2-\sqrt{3}$, being the *smaller*, is the tangent of 15° .

EXAMPLES 16.

Use the formulæ proved in Arts. 49, 50, 52 and 53 to find the values of $\sin(A+B)$, $\sin(A-B)$, $\cos(A+B)$, $\cos(A-B)$, $\tan(A+B)$, and $\tan(A-B)$, for the following values of A and B :

$$(1) \quad A = 60^\circ, \quad B = 30^\circ. \quad (5) \quad A = 30^\circ, \quad B = 15^\circ.$$

$$(2) \quad A = 45^\circ, \quad B = 30^\circ. \quad (6) \quad A = 75^\circ, \quad B = 15^\circ.$$

$$(3) \quad A = 45^\circ, \quad B = 15^\circ. \quad (7) \quad \sin A = \frac{2}{3}, \quad \sin B = \frac{5}{8}.$$

$$(4) \quad A = 60^\circ, \quad B = 15^\circ. \quad (8) \quad \tan A = 5, \quad \tan B = \frac{1}{3}.$$

Write down the values of the following ratios in terms of $\tan A$:—

$$\begin{array}{lll} (9) \quad \tan(45^\circ + A). & (11) \quad \tan(60^\circ - A). & (13) \quad \tan(A - 15^\circ). \\ (10) \quad \tan(45^\circ - A). & (12) \quad \tan(A - 30^\circ). & (14) \quad \cot(A + 75^\circ). \end{array}$$

The Sum and the Difference of two Sines or two Cosines.

56. The formulæ for $\sin(A+B)$, $\sin(A-B)$, $\cos(A+B)$ and $\cos(A-B)$ lead to two very important sets of formulæ which we will now consider.

- (i) $\sin(A+B) + \sin(A-B)$
 $= (\sin A \cos B + \cos A \sin B) + (\sin A \cos B - \cos A \sin B)$
 $= 2 \sin A \cos B.$
- (ii) $\sin(A+B) - \sin(A-B)$
 $= (\sin A \cos B + \cos A \sin B) - (\sin A \cos B - \cos A \sin B)$
 $= 2 \cos A \sin B.$
- (iii) $\cos(A+B) + \cos(A-B)$
 $= (\cos A \cos B - \sin A \sin B) + (\cos A \cos B + \sin A \sin B)$
 $= 2 \cos A \cos B.$
- (iv) $\cos(A-B) - \cos(A+B)$
 $= (\cos A \cos B + \sin A \sin B) - (\cos A \cos B - \sin A \sin B)$
 $= 2 \sin A \sin B.$

If we put $A+B=P$, and $A-B=Q$,
 then $A=\frac{1}{2}(P+Q)$, and $B=\frac{1}{2}(P-Q)$,
 and the above results become

- (i) $\sin P + \sin Q = 2 \sin \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q).$
 (ii) $\sin P - \sin Q = 2 \cos \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q).$
 (iii) $\cos P + \cos Q = 2 \cos \frac{1}{2}(P+Q) \cos \frac{1}{2}(P-Q).$
 (iv) $\cos Q - \cos P = 2 \sin \frac{1}{2}(P+Q) \sin \frac{1}{2}(P-Q).$

These four formulæ must be committed to memory; and particular attention must be paid to the last of them. It will be seen that P being a *greater angle* than Q , it follows that $\cos P$ is a *smaller quantity* than $\cos Q$. The difference between them is therefore $\cos Q - \cos P$, not $\cos P - \cos Q$.

$$\begin{aligned} \text{Example i.} \quad \sin 54^\circ + \sin 13^\circ &= 2 \sin \frac{54^\circ + 13^\circ}{2} \cos \frac{54^\circ - 13^\circ}{2} \\ &= 2 \sin 33\frac{1}{2}^\circ \cos 20\frac{1}{2}^\circ. \end{aligned}$$

$$\begin{aligned} \text{Example ii.} \quad \cos 2A - \cos 5A &= 2 \sin \frac{5A + 2A}{2} \sin \frac{5A - 2A}{2} \\ &= 2 \sin \frac{7A}{2} \sin \frac{3A}{2}. \end{aligned}$$

$$\begin{aligned} \text{Example iii.} \quad \sin 58^\circ - \cos 76^\circ &= \sin 58^\circ - \sin(90^\circ - 76^\circ) \\ &= \sin 58^\circ - \sin 14^\circ \\ &= 2 \cos \frac{58^\circ + 14^\circ}{2} \sin \frac{58^\circ - 14^\circ}{2} \\ &= 2 \cos 36^\circ \sin 22^\circ. \end{aligned}$$

The artifice employed here should be carefully noticed.

EXAMPLES 17.

Express in the form of a product

(1) $\sin 50^\circ + \sin 30^\circ.$	(2) $\sin 20^\circ + \sin 60^\circ.$
(3) $\sin 30^\circ + \sin 40^\circ.$	(4) $\sin 70^\circ + \sin 20^\circ.$
(5) $\sin 30^\circ + \sin 60^\circ.$	(6) $\sin 40^\circ - \sin 30^\circ.$
(7) $\sin 46^\circ - \sin 28^\circ.$	(8) $\sin 59^\circ - \sin 11^\circ.$
(9) $\sin 35^\circ - \sin 23^\circ.$	(10) $\cos 57^\circ + \cos 3^\circ.$
(11) $\cos 28^\circ + \cos 46^\circ.$	(12) $\cos 20^\circ + \cos 30^\circ.$
(13) $\cos 76^\circ + \cos 10^\circ.$	(14) $\cos 34^\circ + \cos 42^\circ.$
(15) $\cos 70^\circ - \cos 80^\circ.$	(16) $\cos 23^\circ - \cos 39^\circ.$
(17) $\cos 36^\circ - \cos 38^\circ.$	(18) $\cos 31^\circ - \cos 59^\circ.$
(19) $\cos 23^\circ - \cos 35^\circ.$	(20) $\cos 3^\circ - \sin 57^\circ.$
(21) $\cos 28^\circ - \sin 42^\circ.$	(22) $\cos 55^\circ - \sin 70^\circ.$
(23) $\cos 46^\circ - \sin 51^\circ.$	(24) $\sin \frac{\pi}{2} + \sin \frac{\pi}{5}.$
(25) $\sin \frac{\pi}{3} - \sin \frac{\pi}{7}.$	(26) $\cos \frac{\pi}{3} + \cos \frac{\pi}{4}.$
(27) $\cos \frac{\pi}{8} - \sin \frac{\pi}{6}.$	(28) $\sin 3A + \sin A.$
(29) $\cos 3A + \cos A.$	(30) $\cos A - \cos 3A.$
(31) $\cos 2A - \cos 3A.$	(32) $\sin 5A + \sin 4A.$
(33) $\sin 5A - \sin 4A.$	(34) $\cos 6A - \cos 9A.$
(35) $\sin (A + B) - \sin (A - B).$	(36) $\cos (A + B) + \cos (A - B).$
(37) $\sin (A + B) + \sin (A - B).$	(38) $\cos (A - B) - \cos (A + B).$
(39) $\cos (n+1)A + \cos (n-1)A.$	(40) $\sin (n+1)A - \sin nA.$

EXAMPLES 18.

Prove that

(1) $\tan A + \tan B = \frac{\sin (A+B)}{\cos A \cos B}.$	(2) $\tan A - \tan B = \frac{\sin (A-B)}{\cos A \cos B}.$
(3) $\tan A + \cot B = \frac{\cos (A-B)}{\cos A \sin B}.$	(4) $\cot B - \cot A = \frac{\sin (A-B)}{\sin A \sin B}.$

Hence reduce to an expression of one term only, without simplifying the result further:—

$$(5) \quad \tan 60^\circ + \tan 30^\circ.$$

$$(6) \quad \tan 60^\circ - \tan 15^\circ.$$

$$(7) \quad \tan 75^\circ - \tan 15^\circ.$$

$$(8) \quad \tan 25^\circ + \tan 40^\circ.$$

$$(9) \quad \cot 15^\circ - \cot 75^\circ.$$

$$(10) \quad \cot 60^\circ - \cot 75^\circ.$$

$$(11) \quad \cot 42^\circ + \tan 26^\circ.$$

$$(12) \quad \cot 28^\circ - \tan 39^\circ.$$

$$(13) \quad \tan 2A + \tan 6A.$$

$$(14) \quad \tan 5A - \tan 3A.$$

$$(15) \quad \cot 2A - \cot 3A.$$

$$(16) \quad \tan A + \cot 3A.$$

The Product of a Sine and Cosine, or two Cosines, or two Sines.

57. We will now consider the formulæ which are the converse of those in Art. 56; namely:—

$$(i) \quad 2 \sin A \cos B = \sin (A + B) + \sin (A - B);$$

$$(ii) \quad 2 \cos A \sin B = \sin (A + B) - \sin (A - B);$$

$$(iii) \quad 2 \cos A \cos B = \cos (A + B) + \cos (A - B);$$

$$(iv) \quad 2 \sin A \sin B = \cos (A - B) - \cos (A + B).$$

These four formulæ should be committed to memory in words. The first and second are very similar. To distinguish them it may be remarked that we have throughout assumed A to be greater than B ; hence, if in the product on the left-hand side the sine is associated with the *greater* angle, the expression is equal to the *sum* of two sines, and if the sine is associated with the *smaller* angle, the expression is equal to the *difference* of two sines.

NOTE. It does not really matter which angle we take to be the greater. For example, if we assume A to be greater than B , we get

$$2 \sin A \cos B = \sin (A + B) + \sin (A - B);$$

and if we assume B to be greater than A , we get

$$2 \sin A \cos B = \sin (B + A) - \sin (B - A).$$

But it will be proved in Art. 81 that $\sin (A - B) = -\sin (B - A)$ for all values of A and B . Hence the two expressions for $2 \sin A \cos B$ are identical.

$$\begin{aligned} \text{Example i.} \quad 2 \sin 50^\circ \cos 38^\circ &= \sin (50^\circ + 38^\circ) + \sin (50^\circ - 38^\circ) \\ &= \sin 88^\circ + \sin 12^\circ. \end{aligned}$$

$$\begin{aligned}\text{Example ii. } 2 \sin 38^\circ \cos 50^\circ &= \sin(50^\circ + 38^\circ) - \sin(50^\circ - 38^\circ) \\ &= \sin 88^\circ - \sin 12^\circ.\end{aligned}$$

$$\begin{aligned}\text{Example iii. } 2 \sin 3A \sin 5A &= \cos(5A - 3A) - \cos(5A + 3A) \\ &= \cos 2A - \cos 8A.\end{aligned}$$

EXAMPLES 19.

Express as the sum or difference of two sines or two cosines :

(1) $2 \sin 50^\circ \cos 17^\circ$.	(2) $2 \sin 61^\circ \cos 8^\circ$.
(3) $2 \sin 33^\circ \cos 28^\circ$.	(4) $2 \sin 79^\circ \cos 10^\circ$.
(5) $2 \sin 27^\circ \cos 6^\circ$.	(6) $2 \cos 55^\circ \sin 30^\circ$.
(7) $2 \cos 71^\circ \sin 14^\circ$.	(8) $2 \cos 20^\circ \sin 19^\circ$.
(9) $2 \cos 86^\circ \sin 3^\circ$.	(10) $2 \cos 34^\circ \sin 19^\circ$.
(11) $2 \cos 49^\circ \cos 36^\circ$.	(12) $2 \cos 59^\circ \cos 12^\circ$.
(13) $2 \cos 46^\circ \cos 18^\circ$.	(14) $2 \cos 25^\circ \cos 24^\circ$.
(15) $2 \cos 8^\circ \cos 6^\circ$.	(16) $2 \sin 43^\circ \sin 40^\circ$.
(17) $2 \sin 6^\circ \sin 2^\circ$.	(18) $2 \sin 24^\circ \sin 29^\circ$.
(19) $2 \sin 78^\circ \sin 3^\circ$.	(20) $2 \sin 32^\circ \sin 5^\circ$.
(21) $2 \sin 8A \cos 3A$.	(22) $2 \sin 4A \cos A$.
(23) $2 \sin 10A \cos 6A$.	(24) $2 \sin 8A \cos 7A$.
(25) $2 \sin 5A \cos 2A$.	(26) $2 \cos 6A \sin A$.
(27) $2 \cos 8A \sin 3A$.	(28) $2 \cos 5A \sin 4A$.
(29) $2 \cos 2A \sin A$.	(30) $2 \cos 7A \sin 5A$.
(31) $2 \cos 8A \cos 2A$.	(32) $2 \cos 7A \cos 4A$.
(33) $2 \cos 6A \cos 5A$.	(34) $2 \sin 3A \sin 2A$.
(35) $2 \sin 9A \sin 8A$.	(36) $2 \cos A \cos B$.
(37) $2 \sin A \sin B$.	(38) $2 \cos A \sin B$.
(39) $2 \cos mA \cos nA$.	(40) $2 \sin mA \sin nB$.

V. THE TRIGONOMETRICAL RATIOS OF $A + B + C$.

58. The formulæ for the sine, cosine, and tangent of $A + B$, may be applied to determine those of $A + B + C$, $A + B + C + D$, &c. in terms of the trigonometrical ratios of A , B , C , D , &c.

As illustrations the following examples will be sufficient ; the results need not be remembered.

$$\begin{aligned}
 \text{(i)} \quad \sin (A + B + C) &= \sin \{(A + B) + C\} \\
 &= \sin (A + B) \cos C + \cos (A + B) \sin C \\
 &= (\sin A \cos B + \cos A \sin B) \cos C \\
 &\quad + (\cos A \cos B - \sin A \sin B) \sin C \\
 &= \sin A \cos B \cos C + \sin B \cos A \cos C \\
 &\quad + \sin C \cos A \cos B - \sin A \sin B \sin C.
 \end{aligned}$$

$$\begin{aligned}
 \text{(ii)} \quad \tan (A + B + C) &= \tan \{A + (B + C)\} \\
 &= \frac{\tan A + \tan (B + C)}{1 - \tan A \tan (B + C)} \\
 &= \frac{\tan A + \frac{\tan B + \tan C}{1 - \tan B \tan C}}{1 - \tan A \frac{\tan B + \tan C}{1 - \tan B \tan C}} \\
 &= \frac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - \tan A \tan B - \tan B \tan C - \tan C \tan A}.
 \end{aligned}$$

EXAMPLES 20.

Find the value of the following in terms of the sine and cosine of A, B and C ;—

$$(1) \quad \sin (A + B - C). \qquad (2) \quad \cos (A + C + B).$$

$$(3) \quad \cos (A - B + C). \qquad (4) \quad \sin (A + B + C + D).$$

If $\tan A = \frac{1}{2}$, $\tan B = \frac{1}{3}$, $\tan C = \frac{1}{4}$, $\tan D = \frac{1}{2}$, find the value of

$$(5) \quad \tan (A + B + C). \qquad (6) \quad \cot (A - B + C).$$

$$(7) \quad \tan (A + B + C + D). \qquad (8) \quad \tan (A + B - C + D).$$

VI. MULTIPLE AND SUB-MULTIPLE ANGLES.

The Trigonometrical Ratios of 2A.

59. *To prove that*

$$\text{(i)} \quad \sin 2A = 2 \sin A \cos A ;$$

$$\begin{aligned}
 \text{(ii)} \quad \cos 2A &= \cos^2 A - \sin^2 A \\
 &= 2 \cos^2 A - 1 \\
 &= 1 - 2 \sin^2 A ;
 \end{aligned}$$

$$\text{(iii)} \quad \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

If in $\sin(A+B)$, $\cos(A+B)$ and $\tan(A+B)$ we put A in place of B , we get $\sin 2A$, $\cos 2A$ and $\tan 2A$.

Thus (i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$;

$$\therefore \sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A.$$

Also (ii) $\cos(A+B) = \cos A \cos B - \sin A \sin B$;

$$\therefore \cos 2A = \cos A \cos A - \sin A \sin A = \cos^2 A - \sin^2 A.$$

Since $\sin^2 A = 1 - \cos^2 A$, and $\cos^2 A = 1 - \sin^2 A$, it follows that this result can also be expressed in either of the following forms:—

$$\cos 2A = 2 \cos^2 A - 1, \text{ or } \cos 2A = 1 - 2 \sin^2 A.$$

Also (iii) $\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$;

$$\therefore \tan 2A = \frac{\tan A + \tan A}{1 - \tan A \tan A} = \frac{2 \tan A}{1 - \tan^2 A}.$$

NOTE. The prefixes *sin*, *cos*, &c. must not be treated as if they were multipliers: for example, $\sin 2A$ must not be considered as equivalent to $2 \sin A$.

60. The formulæ proved in the preceding article must be committed to memory. They connect the sine, cosine and tangent of **any** angle with the ratios of **half** that angle. The student should write out the formulæ for other cases such as $6A$ and $3A$, $5A$ and $\frac{5}{2}A$, &c. The following forms are also very useful in establishing identities:—

$$\left. \begin{aligned} 2 \sin^2 \frac{A}{2} &= 1 - \cos A, \\ 2 \cos^2 \frac{A}{2} &= 1 + \cos A; \end{aligned} \right\} \quad \text{or} \quad \left. \begin{aligned} 1 - \cos A &= 2 \sin^2 \frac{A}{2}, \\ 1 + \cos A &= 2 \cos^2 \frac{A}{2}. \end{aligned} \right\}$$

Also $\tan^2 \frac{A}{2} = \frac{1 - \cos A}{1 + \cos A}.$

61. To prove $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}.$

$$\begin{aligned} \sin 2A &= 2 \sin A \cos A = \frac{2 \sin A \cos A}{\cos^2 A + \sin^2 A} = \frac{2 \times \frac{\sin A}{\cos A}}{1 + \frac{\sin^2 A}{\cos^2 A}} \\ &= \frac{2 \tan A}{1 + \tan^2 A}. \end{aligned}$$

The distinction between the following formulæ must be specially noticed ;

$$\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}, \text{ but } \tan 2A = \frac{2 \tan A}{1 - \tan^2 A}.$$

The Trigonometrical Ratios of $3A$.

62. To prove (i) $\sin 3A = 3 \sin A - 4 \sin^3 A$;

(ii) $\cos 3A = 4 \cos^3 A - 3 \cos A$;

(iii) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$

(i) $\sin 3A = \sin (2A + A)$

$$= \sin 2A \cos A + \cos 2A \sin A$$

$$= 2 \sin A \cos A \cos A + (1 - 2 \sin^2 A) \sin A$$

$$= 2 \sin A \cos^2 A + \sin A - 2 \sin^3 A$$

$$= 2 \sin A (1 - \sin^2 A) + \sin A - 2 \sin^3 A$$

$$= 2 \sin A - 2 \sin^3 A + \sin A - 2 \sin^3 A$$

$$= 3 \sin A - 4 \sin^3 A.$$

(ii) $\cos 3A = \cos (2A + A)$

$$= \cos 2A \cos A - \sin 2A \sin A$$

$$= (2 \cos^2 A - 1) \cos A - 2 \sin A \cos A \sin A$$

$$= 2 \cos^3 A - \cos A - 2 \sin^2 A \cos A$$

$$= 2 \cos^3 A - \cos A - 2 (1 - \cos^2 A) \cos A$$

$$= 2 \cos^3 A - \cos A - 2 \cos A + 2 \cos^3 A$$

$$= 4 \cos^3 A - 3 \cos A.$$

(iii) $\tan 3A = \tan (2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$

$$= \frac{\frac{2 \tan A}{1 - \tan^2 A} + \tan A}{1 - \frac{2 \tan A}{1 - \tan^2 A} \tan A}$$

$$= \frac{2 \tan A + \tan A (1 - \tan^2 A)}{1 - \tan^2 A - 2 \tan^2 A}$$

$$= \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}.$$

These formulæ express the relations which hold between the trigonometrical ratios of any angle and those of $\frac{1}{3}$ of that angle. The student should state them again for such cases as $6A$ and $2A$, A and $\frac{A}{3}$, &c.; and, in order to remember the expressions for $\sin 3A$ and $\cos 3A$ he should notice that

(i) In both cases the numerical coefficients of the terms are 3 and 4, and the indices are 1 and 3; and in both cases the greater coefficient 4 is attached to the term which has the higher index 3; (ii) When the value 0° is put for A , the expression for $\cos 3A$ must reduce to unity. Consequently in $\cos 3A$ the term $4 \cos^3 A$ must come first and $3 \cos A$ second. In $\sin 3A$ this order is *reversed*.

63. *To find the value of $\sin 18^\circ$.*

This may be obtained by means of the formulæ for $\sin 2A$ and $\cos 3A$.

Let $A = 18^\circ$, then $2A + 3A = 90^\circ$;

$$\therefore 2A = 90^\circ - 3A;$$

$$\therefore \sin 2A = \cos 3A;$$

$$\therefore 2 \sin A \cos A = 4 \cos^3 A - 3 \cos A;$$

$$\left. \begin{aligned} \therefore \cos A &= 0, \\ 2 \sin A &= 4 \cos^2 A - 3. \end{aligned} \right\}$$

or

But $\cos 18^\circ$ is certainly not 0;

$$\therefore 2 \sin A = 4 \cos^2 A - 3;$$

$$\therefore 2 \sin A = 4 - 4 \sin^2 A - 3;$$

$$\therefore \sin^2 A + \frac{1}{2} \sin A = \frac{1}{4};$$

$$\therefore \sin A = \frac{-1 \pm \sqrt{5}}{2}.$$

But $\sin 18^\circ$ is certainly not negative; $\therefore \sin 18^\circ = \frac{\sqrt{5}-1}{2}$.

This value should be remembered.

VII. IDENTITIES.

64. We have considered some simple identities in Art. 47. The student should **not** begin by writing down the identity to be proved, and then proceed to simplify **both** members simultaneously, reducing the identity eventually to such a form as $1 = 1$. One of the expressions, usually the left-hand, should be written down by

itself, and modifications should be made in it, step by step, as in the examples below, until the resulting expression is identical in every respect with that on the right-hand.

Under certain circumstances however each expression may be taken *in turn* in the manner described in Art. 40, Example ii.

Example i. Prove that $\sin A \sin 2A + 2 \cos A \cos 2A = 2 \cos^3 A$.

$$\begin{aligned}\text{The expression} &= \sin A \times 2 \sin A \cos A + 2 \cos A (2 \cos^2 A - 1) \\ &= 2 \sin^2 A \cos A + 4 \cos^3 A - 2 \cos A \\ &= 2 \cos A (1 - \cos^2 A) + 4 \cos^3 A - 2 \cos A \\ &= 2 \cos A - 2 \cos^3 A + 4 \cos^3 A - 2 \cos A \\ &= 2 \cos^3 A.\end{aligned}$$

Example ii. Express $\sin^4 A + \cos^4 A$ in terms of $\sin 2A$.

$$\begin{aligned}\sin^4 A + \cos^4 A &= (\sin^2 A + \cos^2 A)^2 - 2 \sin^2 A \cos^2 A \\ &= 1 - \frac{1}{2} (2 \sin A \cos A)^2 \\ &= 1 - \frac{1}{2} \sin^2 2A.\end{aligned}$$

Example iii. Prove that

$$\cos(B+C) \cos(B-C) - \cos(A+C) \cos(A-C) = \sin(A+B) \sin(A-B).$$

$$\begin{aligned}\text{The expression} &= \frac{1}{2} \{2 \cos(B+C) \cos(B-C) - 2 \cos(A+C) \cos(A-C)\} \\ &= \frac{1}{2} \{\cos 2B + \cos 2C - (\cos 2A + \cos 2C)\} \\ &= \frac{1}{2} (\cos 2B - \cos 2A) \\ &= \frac{1}{2} \{2 \sin(A+B) \sin(A-B)\} \\ &= \sin(A+B) \sin(A-B).\end{aligned}$$

Example iv. Prove that $8 \sin^4 A = 3 - 4 \cos 2A + \cos 4A$.

$$\begin{aligned}8 \sin^4 A &= 2 (2 \sin^2 A)^2 \\ &= 2 (1 - \cos 2A)^2 \\ &= 2 (1 - 2 \cos 2A + \cos^2 2A) \\ &= 2 - 4 \cos 2A + 2 \cos^2 2A \\ &= 2 - 4 \cos 2A + 1 + \cos 4A \\ &= 3 - 4 \cos 2A + \cos 4A.\end{aligned}$$

Example v. Prove that $1 + \tan x \tan \frac{x}{2} = \sec x$.

$$\begin{aligned}1 + \tan x \tan \frac{x}{2} &= 1 + \frac{\sin x}{\cos x} \cdot \frac{\sin \frac{x}{2}}{\cos \frac{x}{2}} = \frac{\cos x \cos \frac{x}{2} + \sin x \sin \frac{x}{2}}{\cos x \cos \frac{x}{2}} \\ &= \frac{\cos \left(x - \frac{x}{2}\right)}{\cos x \cos \frac{x}{2}} = \frac{\cos \frac{x}{2}}{\cos x \cos \frac{x}{2}} = \frac{1}{\cos x} \\ &= \sec x.\end{aligned}$$

Example vi. Prove that $\sin x + \sin 2x + \sin 3x = 4 \cos \frac{x}{2} \cos x \sin \frac{3x}{2}$.

$$\begin{aligned}
 \text{The expression} &= (\sin x + \sin 3x) + \sin 2x \\
 &= 2 \sin 2x \cos x + 2 \sin x \cos x \\
 &= 2 \cos x (\sin 2x + \sin x) \\
 &= 2 \cos x \times 2 \sin \frac{3x}{2} \cos \frac{x}{2} \\
 &= 4 \cos \frac{x}{2} \cos x \sin \frac{3x}{2}.
 \end{aligned}$$

Example vii. Prove that

$$4 \sin 3A \sin 5A \sin 7A = \sin A + \sin 5A + \sin 9A - \sin 15A.$$

$$\begin{aligned}
 4 \sin 3A \sin 5A \sin 7A &= 2 \sin 3A (2 \sin 5A \sin 7A) \\
 &= 2 \sin 3A (\cos 2A - \cos 12A) \\
 &= 2 \sin 3A \cos 2A - 2 \sin 3A \cos 12A \\
 &= \sin 5A + \sin A - (\sin 15A - \sin 9A) \\
 &= \sin A + \sin 5A + \sin 9A - \sin 15A.
 \end{aligned}$$

65. When the expression to be transformed involves the second or higher powers of the sine or cosine, these powers can be eliminated by substituting the equivalent expressions involving first powers of the multiple angles. For example, we may put $\frac{1}{2}(1 - \cos 2A)$ for $\sin^2 A$; $\frac{1}{2}(1 + \cos 2A)$ for $\cos^2 A$; $\frac{1}{4}(3 \sin A - \sin 3A)$ for $\sin^3 A$; and $\frac{1}{4}(3 \cos A + \cos 3A)$ for $\cos^3 A$; &c.

Also, in such examples as Nos. 7 to 13 below, the numerator and denominator should be factorised, and the common factors cancelled.

EXAMPLES 21.

Prove that

- (1) $\cos 5A \cos 3A + \sin 5A \sin 3A = \cos 2A.$
- (2) $\cos (A + B) \cos A + \sin A \sin (A + B) = \cos B.$
- (3) $\sin \frac{5A}{2} \cos \frac{A}{2} + \cos \frac{5A}{2} \sin \frac{A}{2} = \sin 3A.$
- (4) $\sin A \cos (B - A) + \cos A \sin (B - A) = \sin B.$
- (5) $\sin (A + B) \sin (A - B) = \sin^2 A - \sin^2 B.$
- (6) $\cos (A + B) \cos (A - B) = \cos^2 B - \sin^2 A.$

- (7) $\frac{\sin 7A - \sin A}{\sin 8A - \sin 2A} = \frac{\cos 4A}{\cos 5A}$. (8) $\frac{\sin 3A + \sin 5A}{\cos 3A - \cos 5A} = \cot A$.
- (9) $\frac{\sin 3A - \sin A}{\cos A - \cos 3A} = \cot 2A$. (10) $\frac{\cos 7A - \cos 9A}{\sin 9A - \sin 7A} = \tan 8A$.
- (11) $\frac{\sin 7A + \sin 5A}{\cos 5A - \cos 7A} = \cot A$. (12) $\frac{\sin A + \sin B}{\cos A + \cos B} = \tan \frac{A+B}{2}$.
- (13) $\frac{\sin A + \sin 5A}{\sin 4A + \sin 2A} = 2 \cos A - \sec A$.
- (14) $1 + \tan A \tan 2A = \sec 2A$.
- (15) $1 + \cot A \cot \frac{A}{2} = \operatorname{cosec} A \cot \frac{A}{2}$.
- (16) $\frac{\tan(A+B) + \tan(A-B)}{\tan(A+B) - \tan(A-B)} = \frac{\sin 2A}{\sin 2B}$.
- (17) $(\cos A + \sin A)^2 = 1 + \sin 2A$.
- (18) $(\cos A - \sin A)^2 = 1 - \sin 2A$.
- (19) $\operatorname{cosec} 2A = \frac{1}{2} \operatorname{cosec} A \sec A$.
- (20) $1 - \frac{1}{2} \sin 2A \tan A = \cos^2 A$.

EXAMPLES 22.

Prove that

- (1) $\sin A \cos^3 A + \cos A \sin^3 A = \frac{1}{2} \sin 2A$.
- (2) $\cos^4 A - \sin^4 A = \cos 2A$.
- (3) $\sin A \cos^5 A - \cos A \sin^5 A = \frac{1}{4} \sin 4A$.
- (4) $\cos 4A = 8 \cos^4 A - 8 \cos^2 A + 1$.
- (5) $\cot \frac{A}{2} + \tan \frac{A}{2} = 2 \operatorname{cosec} A$.
- (6) $\sin^2 A \cos^3 A = \frac{1}{8} (1 - \cos 4A)$.
- (7) $\cos^4 A + \sin^4 A = \frac{1}{4} (3 + \cos 4A)$.
- (8) $\tan 4A = \frac{4 \tan A - 4 \tan^3 A}{1 - 6 \tan^2 A + \tan^4 A}$.
- (9) $\tan(45^\circ + A) - \tan(45^\circ - A) = 2 \tan 2A$.
- (10) $\tan 3A - \tan A = 2 \sin A \sec 3A$.
- (11) $\frac{\sin 2A}{1 + \cos 2A} = \tan A$. (12) $\sqrt{\frac{2 \sec A}{1 + \sec A}} = \sec \frac{A}{2}$.

$$(13) \quad \operatorname{cosec} 2A + \cot 2A = \cot A.$$

$$(14) \quad \frac{2 \sin A - \sin 2A}{2 \sin A + \sin 2A} = \tan^2 \frac{A}{2}. \quad (15) \quad \frac{\sin 3A}{\sin A} = 1 + 2 \cos 2A.$$

$$(16) \quad \sin^3 A \cos^3 A = \frac{1}{32} (3 \sin 2A - \sin 6A).$$

$$(17) \quad 4 (\cos^3 A \sin 3A + \sin^3 A \cos 3A) = 3 \sin 4A.$$

$$(18) \quad \frac{1 + \sin A - \cos A}{1 + \sin A + \cos A} = \tan \frac{A}{2}.$$

$$(19) \quad (\sin A - \sin B)^2 + (\cos A - \cos B)^2 = 2 \operatorname{versin} (A - B).$$

$$(20) \quad \frac{1 + \tan 2A \tan A}{\tan A + \cot A} = \frac{1}{2} \tan 2A.$$

EXAMPLES 23.

Prove that

$$(1) \quad \sin (A + B + C) + \sin (A + B - C) + \sin (A - B + C) \\ + \sin (A - B - C) = 4 \sin A \cos B \cos C.$$

$$(2) \quad \cos (A + B + C) + \cos (A + B - C) + \cos (A - B + C) \\ + \cos (A - B - C) = 4 \cos A \cos B \cos C.$$

$$(3) \quad \sin 2A + \sin 4A + \sin 6A = 4 \cos A \cos 2A \sin 3A.$$

$$(4) \quad \sin A \cos (B - C) - \sin B \cos (A - C) = \cos C \sin (A - B).$$

$$(5) \quad 4 \cos A \cos B \cos C = \cos (A - B - C) + \cos (A + B + C) \\ + \cos (A - B + C) + \cos (A + B - C).$$

$$(6) \quad 4 \cos A \cos 2A \cos 3A = \cos 6A + \cos 4A + \cos 2A + 1.$$

$$(7) \quad \frac{\sin A + \sin 2A + \sin 3A}{\cos A + \cos 2A + \cos 3A} = \tan 2A.$$

$$(8) \quad \frac{\sin A + 2 \sin 3A + \sin 5A}{\sin 3A + 2 \sin 5A + \sin 7A} = \frac{\sin 3A}{\sin 5A}.$$

EXAMPLES 24. MISCELLANEOUS.

(1) Find the circular measure of an angle of a regular polygon of 16 sides.

(2) If 147 square feet be represented by the number 3, what is the unit of linear measurement?

(3) If 5000 cubic inches be represented by the number 40, what is the unit of linear measurement?

(4) Express $\frac{3}{7}$ of a radian in sexagesimal measure. ($\pi = \frac{22}{7}$.)

(5) The angles of a triangle are $4x$ degrees, $5x$ degrees and $10x$ grades. Find x , and express each angle in degrees.

- (6) If $\sec A = \frac{5}{3}$, find the sine, cosine and tangent of A .
- (7) If $\operatorname{cosec} A = \frac{1}{5x}$, and $\sin B = \frac{y}{4}$, what are the greatest numerical values of x and y ?
- (8) Write down the complement of $36^\circ 41' 57''$, and of $\frac{3\pi}{7}$.
- (9) Construct an angle whose cosecant is $2\sqrt{3}$.
- (10) Construct an angle whose cotangent is $\frac{1}{2}\sqrt{5}$.
- (11) A chord, 9 inches long, is 6 inches from the centre of a circle; find the length of the diameter of the circle.
- (12) Prove $1 - \tan^4 A = 2 \sec^2 A - \sec^4 A$.
- (13) Prove $\tan A + \cot A = \sqrt{\sec^2 A + \operatorname{cosec}^2 A}$.
- (14) Prove $\cos^6 A + \sin^6 A = 1 - 3 \cos^2 A \sin^2 A$.
- (15) Prove $\sec^6 A - \tan^6 A = 1 + 3 \tan^2 A \sec^2 A$.
- (16) Prove $\frac{\sin 5^\circ + \sin 47^\circ}{\cos 5^\circ - \cos 47^\circ} = \tan 69^\circ$.
- (17) Prove $\frac{\cos 35^\circ - \sin 35^\circ}{\cos 35^\circ + \sin 35^\circ} = \tan 10^\circ$.
- (18) Prove $\tan A \cot \frac{A}{2} - 1 = \sec A$.
- (19) Prove $\frac{\tan 2A - \tan A}{\tan A} = \sec 2A$.
- (20) Prove $\frac{1 - \tan^2 (45^\circ - A)}{1 + \tan^2 (45^\circ - A)} = \sin 2A$.
- (21) Prove $\cos 40^\circ - \cos 20^\circ + \cos 80^\circ = 0$.
- (22) Prove $\frac{\tan 3A + \tan A}{\tan 3A - \tan A} = 2 \cos 2A$.
- (23) Prove $\frac{\cos 3A + \sin 3A}{\cos A - \sin A} = 1 + 4 \sin A \cos A$.
- (24) Prove $\frac{\cos A}{1 + \cos A} \times \frac{\sin 2A}{1 + \cos 2A} = \tan \frac{A}{2}$.
- (25) As seen from an opposite window, the top of a house has an elevation of 60° , and the height of the house subtends a right angle. If the width of the street be 30 feet, find the height of the house.

VIII. ANGLES GREATER THAN A RIGHT ANGLE AND NEGATIVE ANGLES.

66. We have hitherto considered the trigonometrical ratios of only such angles as are equal to, or *less than*, a right angle. We will now proceed to the consideration of angles *greater than* a right angle.

If through a point O two straight lines XOX' and YOY' be drawn at right angles to one another, and of indefinite length, these lines will divide the space round the point O into four parts called **Quadrants**.

The space between OX and OY , produced indefinitely, is called the **first quadrant**; that between OY and OX' is called the **second quadrant**; that between OX' and OY' is called the **third quadrant**; and that between OY' and OX is called the **fourth quadrant**.

Also, OX is called the **Initial Line**.

We may now consider an angle to be described by a line, which we will call the **radius**, starting from coincidence with OX , and turning about the extremity O in the direction contrary to that of the hands of a clock, i.e., *contra-clockwise*.

(i) If the angle to be described be less than one right angle, the radius will stop in its course somewhere between the position OX and the position OY , i.e., somewhere in the first quadrant. An angle less than a right angle, such as XOP , may therefore be spoken of as *an angle in the first quadrant*.

(ii) If the angle described be equal to one right angle, the radius will stop in the position OY .

(iii) If the angle described lie in magnitude between one right angle and two right angles, the radius will stop somewhere between the position OY and the position OX' ; such an angle as XOQ may therefore be said to be *in the second quadrant*.

(iv) If the angle described be equal to two right angles, the radius will stop in the position OX' .

(v) If the angle described lie in magnitude between two right angles and three right angles, the radius will stop somewhere between the position OX' and the position OY' ; such an angle as XOR may therefore be said to be *in the third quadrant*.

(vi) If the angle described be equal to three right angles, the radius will stop in the position OY' .

(vii) If the angle described lie in magnitude between three right angles and four right angles, the radius will stop in its course somewhere between the position OY' and the position OX ; such an angle as XOS may therefore be said to be *in the fourth quadrant*.

(viii) If the angle described be equal to four right angles, the radius will stop on reaching the position OX from which it started.

(ix) If the angle described be greater than four right angles, the radius will proceed on for a second revolution, and, if necessary, for a third or more.

67. The radius might however turn in the other direction, or *clockwise*, reaching OY' first, then OX' , OY and OX , in that order. In the former case it is said to revolve in the **positive** direction, and an angle so described is a **positive** angle; while in the latter case it is said to turn in the **negative** direction, and an angle described by it is a **negative** angle.

68. Again, any position of the radius, such as OQ in the diagram, would be arrived at in the course of the first revolution, either clockwise or contra-clockwise; again in the second revolution, and once again in each successive revolution.

For example, if OQ be midway between OY and OX' , it will be reached when the radius has turned *contra-clockwise* through any one of the angles 135° , $360^\circ + 135^\circ$, $720^\circ + 135^\circ$,

and, generally, through $n \times 360^\circ + 135^\circ$, where n stands for any positive integer; or *clockwise* through any one of the angles -225° , $-360^\circ - 225^\circ$, $-720^\circ - 225^\circ$, and, generally, through $-n \times 360^\circ - 225^\circ$, where n again stands for any positive integer.

Positive and Negative Lines.

69. A straight line may be considered as generated by the motion of a point which moves from one extremity of it to the other.

For example, if a point start from the position A and move to the *right* directly towards the position B, it will describe the straight line AB. If, on the other $\overline{\text{A} \qquad \text{B}}$ hand, the point were to start from the position B and move to the *left* directly towards the position A, it would describe the same straight line, but in the opposite direction. In the latter case its path would be referred to as the line BA, not as the line AB.

Consequently, when we consider a straight line as generated by the motion of a point, the direction of motion is inferred from the order in which the letters are stated; AB denoting that the motion is from A to B, and BA that the motion is from B to A.

70. If a point move from A to B, and then back from B to A, the final position of the point is the same as the first. Consequently, there being in the end no change of position, we may say that

$$AB + BA = 0, \text{ or } AB = -BA.$$

Thus we see that in the case of a straight line considered as generated by the motion of a point, a line described by motion from left to right may be called **positive**, and a line described by motion from right to left, **negative**.

On this hypothesis the line AB is *positive* and the line BA is *negative*.

The Trigonometrical Ratios.

71. In defining the Trigonometrical Ratios of an angle we took a point P in one of the lines forming the angle, and from it we drew PM perpendicular to the other. For simplicity we took the point P in the *revolving* line, and we drew the perpendicular PM upon the initial line OX. It will now be seen that,

according to the size of the angle, the point **M** may fall either in **OX**, or in **XO** produced, i.e., either to the *right* of **O**, or to the *left* of **O**.

The point **M** will fall to the *right* of **O**, when the revolving line is in the 1st or in the 4th quadrant (Figs. i. and iv.); and it will fall to the *left* of **O**, when the revolving line is in the 2nd or in the 3rd quadrant (Figs. ii. and iii.).

In each of these four cases, $\frac{MP}{OP}$, $\frac{OM}{OP}$ and $\frac{MP}{OM}$ are respectively the sine, cosine and tangent of the angle **XOP**, in accordance with the definition (Art. 26).

To distinguish between the cases it is agreed, in accordance with Art. 70, that distances measured along the initial line **OX**, from **O** towards the *right*, shall be considered **positive**, and those measured in the opposite direction, from **O** towards the *left*, **negative**. It is also agreed that lines measured along or parallel to the upright **OY**, in the direction from **O** to **Y**, shall be considered **positive**, and lines measured in the opposite direction, from **O** towards **Y'**, **negative**.

For example, in (i) **OM** and **MP** are both positive ;

in (ii) **OM** is negative, and **MP** positive ;

in (iii) **OM** and **MP** are both negative ;

in (iv) **OM** is positive, and **MP** negative.

The distance **OP**, measured along the revolving line from **O** outwards, is always taken as **positive**.

72. In all four cases $\frac{MP}{OP}$ is the *sine* of the angle **XOP**; hence we see that the *sine* is **positive** for any angle in the **1st** or **2nd**, and **negative** for any angle in the **3rd** or **4th** quadrant.

Again, in all four cases, $\frac{OM}{OP}$ is the *cosine* of the angle **XOP**; hence the *cosine* is **positive** for any angle in the **1st** or **4th**, and **negative** for any angle in the **2nd** or **3rd** quadrant.

Again, in all four cases, $\frac{MP}{OM}$ is the *tangent* of the angle **XOP**; hence the *tangent* is **positive** for any angle in the **1st** or **3rd**, and **negative** for any angle in the **2nd** or **4th** quadrant.

We may also state these facts as follows:—

(i) For an angle in the **1st** quadrant, all the **trigonometrical ratios** are positive ;

(ii) For an angle in the **2nd** quadrant, only the **sine** and **cosecant** are positive ;

(iii) For an angle in the **3rd** quadrant, only the **tangent** and **cotangent** are positive ;

(iv) For an angle in the **4th** quadrant, only the **cosine** and **secant** are positive.

73. It must be noticed that in all cases one of the lines bounding the angle is the initial line **OX**, and that the angle considered is **always XOP**, and **never X'OP**.

The Trigonometrical Ratios of 180° .

74. Let OB be a position of the revolving line as it approaches the position OX' . Take any point P in it, and draw PM perpendicular to OX' .

As OB moves up to coincidence with OX' , MP decreases to zero, and OM increases until it becomes equal in magnitude to OP .

But OM is negative, and OP is positive; consequently, when OB coincides with OX' , we have

$$OM = -OP.$$

Hence

$$\begin{aligned}\sin 180^\circ &= \frac{MP}{OP} = \frac{0}{OP} = 0, & \operatorname{cosec} 180^\circ &= \frac{OP}{MP} = \frac{OP}{0} = \infty, \\ \cos 180^\circ &= \frac{OM}{OP} = \frac{-OP}{OP} = -1, & \sec 180^\circ &= \frac{OP}{OM} = \frac{OP}{-OP} = -1, \\ \tan 180^\circ &= \frac{MP}{OM} = \frac{0}{-OP} = 0, & \cot 180^\circ &= \frac{OM}{MP} = \frac{-OP}{0} = -\infty.\end{aligned}$$

The Trigonometrical Ratios of 270° .

75. Let OB be a position of the revolving line as it approaches the position OY' . Take any point P in it, and draw PM perpendicular to OX' .

As OB moves up to coincidence with OY' , OM decreases to zero, and MP increases until it becomes equal in magnitude to OP .

But MP is negative, and OP is positive; consequently, when OB coincides with OY' , we have

$$MP = -OP.$$

Hence

$$\sin 270^\circ = \frac{MP}{OP} = \frac{-OP}{OP} = -1, \quad \operatorname{cosec} 270^\circ = \frac{OP}{MP} = \frac{OP}{-OP} = -1,$$

$$\cos 270^\circ = \frac{OM}{OP} = \frac{0}{OP} = 0, \quad \sec 270^\circ = \frac{OP}{OM} = \frac{OP}{0} = \infty,$$

$$\tan 270^\circ = \frac{MP}{OM} = \frac{-OP}{0} = -\infty, \quad \cot 270^\circ = \frac{OM}{MP} = \frac{0}{-OP} = 0.$$

NOTE. The student should endeavour to recall the values of the ratios of 180° and 270° , whenever he needs them, by means of a **mental picture**. He should not attempt to commit the values to memory as so many disconnected quantities.

EXAMPLES 25.

In each of the following cases state (with a diagram) the quadrant in which the revolving line lies; and say also in each case which of the trigonometrical ratios are negative.

(1) 75° .	(2) 127° .	(3) 62° .	(4) 131° .
(5) 163° .	(6) 252° .	(7) 25° .	(8) 330° .
(9) 305° .	(10) 179° .	(11) 271° .	(12) 199° .
(13) 287° .	(14) 269° .	(15) 145° .	(16) 359° .
(17) -5° .	(18) -269° .	(19) -330° .	(20) -89° .
(21) 260° .	(22) 15° .	(23) -15° .	(24) 200° .
(25) $\frac{3\pi}{5}$.	(26) $-\frac{3\pi}{5}$.	(27) $-\frac{215\pi}{7}$.	(28) $\frac{320\pi}{3}$.

In the following examples, n represents an integer.

(29) $2n\pi + \frac{\pi}{4}$.	(30) $2n\pi - \frac{\pi}{3}$.	(31) $(2n+1)\pi - \frac{\pi}{3}$.
(32) $-2n\pi + \frac{2\pi}{3}$.	(33) $(2n-1)\pi + \frac{5\pi}{12}$.	(34) $-2n\pi - \frac{2\pi}{3}$.

The Complement and the Supplement of an Angle.

76. In Art. 29 we said that one angle is called the **complement** of another, when the sum of the two angles is a right angle. This definition has now a somewhat wider meaning, as will be seen from the examples below. The word *sum* will now be understood to mean *algebraic sum*.

- (i) The complement of $33^\circ 15' 24'' = 90^\circ - 33^\circ 15' 24''$
 $= 56^\circ 44' 36''$.
- (ii) The complement of $\frac{\pi}{7} = \frac{\pi}{2} - \frac{\pi}{7} = \frac{5\pi}{14}$.
- (iii) The complement of $-220^\circ = 90^\circ - (-220^\circ) = 310^\circ$.
- (iv) The complement of $130^\circ = 90^\circ - 130^\circ = -40^\circ$.

77. DEFINITION. One angle is said to be the **supplement** of another, when the algebraic **sum** of the two angles is **two** right angles.

- (i) The supplement of $23^\circ 5' 14'' = 180^\circ - 23^\circ 5' 14''$
 $= 156^\circ 54' 46''$.
- (ii) The supplement of $-\frac{3\pi}{5} = \pi - \left(-\frac{3\pi}{5}\right) = \frac{8\pi}{5}$.
- (iii) The supplement of $315^\circ = 180^\circ - 315^\circ = -135^\circ$.

EXAMPLES 26.

Write down (i) the complement, (ii) the supplement, of each of the following angles.

(1) 23° .	(2) $19^\circ 8' 21''$.	(3) -49° .
(4) $\frac{\pi}{8}$.	(5) $\frac{3\pi}{7}$.	(6) $-\frac{\pi}{8}$.
(7) $-37^\circ 58'$.	(8) $-83^\circ 14' 55''$.	(9) $-31^\circ 5' 7''$.
(10) 218° .	(11) $195^\circ 42'$.	(12) $225^\circ 33' 54''$.
(13) $\frac{5\pi}{3}$.	(14) $\frac{21\pi}{16}$.	(15) $-\frac{7\pi}{6}$.
(16) -420° .	(17) $\pi + A$.	(18) $\frac{2\pi}{3} - A$.

78. We will now use the conventions in respect of algebraic sign to determine the relations which the trigonometrical ratios of the angles $180^\circ - A$, $180^\circ + A$, $360^\circ - A$, and $-A$ bear to the trigonometrical ratios of the angle A . The relations will be true for all cases, and the method of proof will be the same whatever be the magnitude of A , but in drawing the figure we will assume that the angle A is in the first quadrant. We will assume also that A expresses the magnitude of the angle in degrees.

The Trigonometrical Ratios of $180^\circ - A$.

79. Let OA be the position of the revolving line when, having started from the position OX , it has described the angle A ; and OB its position when it has described the angle $180^\circ - A$.

Thus $\hat{XOA} = A$,
and $\hat{XOB} = 180^\circ - A$.

In OA and OB take points P and P' respectively, such that $OP' = OP$; and from P and P' draw PM and $P'M'$ perpendicular respectively to OX and XO produced.

Then in the triangles MOP , $M'OP'$ we have

$$OP = OP', \quad \hat{OMP} = \hat{OM'P'}, \quad \text{and} \quad \hat{MOP} = \hat{M'OP'};$$

hence, by Euclid I. 26, the triangles MOP and $M'OP'$ are equal in all respects;

$$\left. \begin{aligned} \therefore \sin(180^\circ - A) &= \sin XOB = \frac{M'P'}{OP'} = \frac{MP}{OP} = \sin A, \\ \cos(180^\circ - A) &= \cos XOB = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A, \\ \text{and } \tan(180^\circ - A) &= \tan XOB = \frac{M'P'}{OM'} = \frac{MP}{-OM} = -\tan A. \end{aligned} \right\}$$

$$\text{Similarly, } \cot(180^\circ - A) = -\cot A,$$

$$\sec(180^\circ - A) = -\sec A,$$

$$\text{and } \operatorname{cosec}(180^\circ - A) = \operatorname{cosec} A.$$

The Trigonometrical Ratios of $180^\circ + A$.

80. Let OA be the position of the revolving line when, having started from the position OX , it has described the angle A ; and let OB be its position when it has described the angle $180^\circ + A$. Thus

$$\hat{XOA} = A,$$

$$\text{and } \hat{XOB} = 180^\circ + A.$$

In OA and OB take points P and P' respectively, such that $OP' = OP$; and from P and P' draw PM and $P'M'$ perpendicular respectively to OX and XO produced.

Then in the triangles MOP , $M'OP'$ we have

$$OP = OP', \quad \angle OMP = \angle OM'P', \quad \text{and} \quad \angle MOP = \angle M'OP';$$

hence, by Euclid I. 26, the triangles MOP and $M'OP'$ are equal in all respects;

$$\left. \begin{aligned} \therefore \sin(180^\circ + A) &= \sin XOB = \frac{M'P'}{OP'} = \frac{-MP}{OP} = -\sin A, \\ \cos(180^\circ + A) &= \cos XOB = \frac{OM'}{OP'} = \frac{-OM}{OP} = -\cos A, \\ \text{and } \tan(180^\circ + A) &= \tan XOB = \frac{M'P'}{OM'} = \frac{-MP}{-OM} = \frac{MP}{OM} = \tan A. \end{aligned} \right\}$$

Similarly, $\cot(180^\circ + A) = \cot A,$
 $\sec(180^\circ + A) = -\sec A,$
 and $\operatorname{cosec}(180^\circ + A) = -\operatorname{cosec} A.$

The Trigonometrical Ratios of $360^\circ - A$ and $-A$.

81. Let OA be the position of the revolving line when, having started from the position OX , it has described the angle A ; and let OB be its position when it has described the angle

$$360^\circ - A;$$

OB will be its position also when, revolving in the *negative* direction, it has described the angle $-A$.

Thus

$$\angle XOA = A,$$

and

$$\angle XOB = 360^\circ - A \text{ or } -A.$$

In OA and OB take points P and P' respectively, such that $OP' = OP$; and draw PP' meeting OX at M .

Then in the triangles MOP , MOP' we have

$$OP = OP', \quad \hat{MOP} = \hat{MOP'}, \quad \text{and} \quad \hat{MPO} = \hat{MP'O};$$

hence, by Euclid I. 26, the triangles MOP and MOP' are equal in all respects, and are therefore right-angled at M ;

$$\left. \begin{array}{l} \therefore \sin(360^\circ - A) \\ \text{or} \quad \sin(-A) \end{array} \right\} = \sin XOB = \frac{MP'}{OP'} = \frac{-MP}{OP} = -\sin A, \\ \left. \begin{array}{l} \cos(360^\circ - A) \\ \text{or} \quad \cos(-A) \end{array} \right\} = \cos XOB = \frac{OM}{OP'} = \frac{OM}{OP} = \cos A, \\ \left. \begin{array}{l} \tan(360^\circ - A) \\ \text{or} \quad \tan(-A) \end{array} \right\} = \tan XOB = \frac{MP'}{OM} = \frac{-MP}{OM} = -\tan A.$$

$$\left. \begin{array}{l} \text{Similarly,} \quad \cot(360^\circ - A) = \cot(-A) = -\cot A, \\ \quad \sec(360^\circ - A) = \sec(-A) = \sec A, \\ \quad \operatorname{cosec}(360^\circ - A) = \operatorname{cosec}(-A) = -\operatorname{cosec} A. \end{array} \right\}$$

82. The results of Arts. 79—81 may be summed up as follows:—The Trigonometrical ratios of $180^\circ - A$, $180^\circ + A$, $360^\circ - A$, $-A$, and of these angles increased by any multiple of 360° , are numerically the same as the corresponding ratios of the angle A , but, in certain cases depending upon the quadrant in which the revolving line lies, some of the ratios are algebraically negative.

The student should not attempt to commit these results to memory separately. He need only picture mentally the position of the revolving line to ascertain any algebraic sign at once.

EXAMPLES 27.

State the sine, cosine and tangent of

(1) $180^\circ - 45^\circ$.	(2) $180^\circ + 45^\circ$.	(3) $360^\circ - 45^\circ$.
(4) $180^\circ - 30^\circ$.	(5) $180^\circ + 30^\circ$.	(6) $360^\circ - 30^\circ$.
(7) $\pi - \frac{\pi}{3}$.	(8) $\pi + \frac{\pi}{3}$.	(9) $2\pi - \frac{\pi}{3}$.
(10) $180^\circ - 75^\circ$.	(11) $180^\circ + 75^\circ$.	(12) $360^\circ - 75^\circ$.
(13) -45° .	(14) -30° .	(15) -60° .
(16) -15° .	(17) -75° .	(18) $-\frac{\pi}{2}$.

State the sine of

(19) $180^\circ - 18^\circ$.	(20) $180^\circ + 18^\circ$.	(21) $360^\circ - 18^\circ$.

83. We will now give a few examples to show how the trigonometrical ratios of any angle however large may be expressed in terms of the trigonometrical ratios of an angle in the first quadrant, whence their values can be at once ascertained.

Example i. Determine the trigonometrical ratios of 135° .

$$\sin 135^\circ = \sin (180^\circ - 45^\circ) = \sin 45^\circ = \frac{1}{\sqrt{2}},$$

$$\cos 135^\circ = \cos (180^\circ - 45^\circ) = -\cos 45^\circ = -\frac{1}{\sqrt{2}},$$

$$\tan 135^\circ = \tan (180^\circ - 45^\circ) = -\tan 45^\circ = -1; \text{ \&c.}$$

Example ii. Determine the trigonometrical ratios of $\frac{7\pi}{6}$.

$$\sin \frac{7\pi}{6} = \sin \left(\pi + \frac{\pi}{6} \right) = -\sin \frac{\pi}{6} = -\frac{1}{2},$$

$$\cos \frac{7\pi}{6} = \cos \left(\pi + \frac{\pi}{6} \right) = -\cos \frac{\pi}{6} = -\frac{\sqrt{3}}{2};$$

$$\tan \frac{7\pi}{6} = \tan \left(\pi + \frac{\pi}{6} \right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}; \text{ \&c.}$$

Example iii. Determine the trigonometrical ratios of 840° .

$$\sin 840^\circ = \sin (720^\circ + 120^\circ) = \sin 120^\circ = \sin (180^\circ - 60^\circ) = \sin 60^\circ = \frac{\sqrt{3}}{2},$$

$$\cos 840^\circ = \cos (720^\circ + 120^\circ) = \cos 120^\circ = \cos (180^\circ - 60^\circ) = -\cos 60^\circ = -\frac{1}{2},$$

$$\tan 840^\circ = \tan (720^\circ + 120^\circ) = \tan 120^\circ = \tan (180^\circ - 60^\circ) = -\tan 60^\circ = -\sqrt{3};$$

\&c.

EXAMPLES 28.

Determine the sine, cosine and tangent of the following angles.

(1) 150° .	(2) 120° .	(3) 240° .	(4) 225° .	(5) 315° .
(6) 330° .	(7) 105° .	(8) 165° .	(9) 195° .	(10) 255° .
(11) 480° .	(12) 495° .	(13) 510° .	(14) 585° .	(15) 555° .
(16) $\frac{2\pi}{3}$.	(17) $\frac{7\pi}{4}$.	(18) $\frac{5\pi}{3}$.	(19) $\frac{7\pi}{6}$.	(20) $\frac{11\pi}{2}$.

Determine the following trigonometrical ratios.

(21) $\sin 405^\circ$.	(22) $\cos 390^\circ$.	(23) $\tan 420^\circ$.
(24) $\sin 378^\circ$.	(25) $\tan 375^\circ$.	(26) $\cos 750^\circ$.
(27) $\sin 780^\circ$.	(28) $\cot 765^\circ$.	(29) $\cos 840^\circ$.
(30) $\sin 930^\circ$.	(31) $\sec 690^\circ$.	(32) $\operatorname{cosec} 660^\circ$.
(33) $\cot 585^\circ$.	(34) $\cos 870^\circ$.	(35) $\sin 945^\circ$.

84. What has been said with regard to *positive* angles applies also to *negative* angles, i.e., to angles which have been described by the revolving line moving in the *negative*, or clock-wise, direction; for both the numerical value and the algebraic sign of a ratio depend solely upon the position of the revolving line, and it is quite immaterial whether this position has been attained by motion in the positive direction, or by motion in the negative direction.

To determine a trigonometrical ratio in the case of a negative angle, it is often convenient to add 4 right angles to the angle, or such a multiple of 4 right angles as will make a positive result. None of the ratios will be affected by any such addition.

$$\text{Example i. } \tan(-330^\circ) = \tan(360^\circ - 330^\circ) = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

$$\begin{aligned} \text{Example ii. } \sin(-840^\circ) &= \sin(3 \times 360^\circ - 840^\circ) = \sin 240^\circ \\ &= \sin(180^\circ + 60^\circ) = -\sin 60^\circ \\ &= -\frac{\sqrt{3}}{2}. \end{aligned}$$

$$\begin{aligned} \text{Example iii. } \tan(-3\frac{1}{3}\pi) &= \tan(4\pi - 3\frac{1}{3}\pi) = \tan \frac{2\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right) \\ &= -\tan \frac{\pi}{3} \\ &= -\sqrt{3}. \end{aligned}$$

EXAMPLES 29.

Find the values of the sine, cosine and tangent of

(1) -210° .	(2) -405° .	(3) -420° .	(4) -195° .
(5) -285° .	(6) -135° .	(7) -120° .	(8) -150° .
(9) -165° .	(10) -105° .	(11) -240° .	(12) -225° .
(13) -330° .	(14) -315° .	(15) -300° .	(16) -345° .
(17) $-\frac{\pi}{4}$.	(18) $-\frac{\pi}{3}$.	(19) $-\frac{5\pi}{6}$.	(20) $-\frac{7\pi}{6}$.

Determine the values of the following trigonometrical ratios.

(21) $\cos(-765^\circ)$.	(22) $\sin(-390^\circ)$.	(23) $\tan(-780^\circ)$.
(24) $\sin(-378^\circ)$.	(25) $\tan(-375^\circ)$.	(26) $\sec(-660^\circ)$.
(27) $\operatorname{cosec}(-690^\circ)$.	(28) $\cos(-930^\circ)$.	(29) $\cot(-840^\circ)$.
(30) $\cot(-1020^\circ)$.	(31) $\tan(-1305^\circ)$.	(32) $\cos(-1200^\circ)$.
(33) $\sec(-7\frac{2}{3}\pi)$.	(34) $\operatorname{cosec}(-5\frac{3}{4}\pi)$.	(35) $\cot(-15\frac{5}{6}\pi)$.

The Trigonometrical Ratios of $90^\circ - A$.

85. In drawing the figure we will, as in Art. 78, assume that the angle A is in the first quadrant, but the method of proof will be the same, and the results will be equally true, whatever the magnitude of A may be. We assume also that the symbol A expresses the measure of the angle in degrees.

Let OA be the position of the revolving line when, having started from the position OX , it has described the angle A ; and let OB be its position when it has described the angle $90^\circ - A$.

Thus $\hat{XOA} = A$,

and $\hat{XOB} = 90^\circ - A$.

In OA and OB take points P and P' respectively, such that $OP' = OP$; and from P and P' draw PM and $P'M'$ perpendicular to OX . Then in the triangles MOP and $M'P'O$ we have

$$OP = OP', \quad \hat{PMO} = \hat{OM'P'}, \quad \text{and} \quad \hat{MPO} = 90^\circ - A = \hat{M'OP'};$$

hence, by Euclid I. 26, the triangles MOP and $M'P'O$ are equal in all respects;

$$\therefore OM = M'P', \quad \text{and} \quad MP = OM';$$

$$\left. \begin{aligned} \therefore \sin(90^\circ - A) &= \sin \angle XOB = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos A, \\ \cos(90^\circ - A) &= \cos \angle XOB = \frac{OM'}{OP'} = \frac{MP}{OP} = \sin A, \\ \tan(90^\circ - A) &= \tan \angle XOB = \frac{M'P'}{OM'} = \frac{OM}{MP} = \cot A; \end{aligned} \right\}$$

$$\text{and similarly,} \quad \left. \begin{aligned} \cot(90^\circ - A) &= \tan A, \\ \sec(90^\circ - A) &= \operatorname{cosec} A, \\ \operatorname{cosec}(90^\circ - A) &= \sec A. \end{aligned} \right\}$$

The Trigonometrical Ratios of $90^\circ + A$.

86. Let OA be the position of the revolving line when, having started from the position OX , it has described the angle A ; and let OB be its position when it has described the angle $90^\circ + A$.

Thus $\hat{XOA} = A$,
and $\hat{XOB} = 90^\circ + A$.

In OA and OB take points P and P' respectively, such that $OP' = OP$; and from P and P' draw PM and $P'M'$ perpendicular to $X'OX$. Then in the triangles MOP and $M'P'O$ we have

$$OP = OP', \quad \hat{PMO} = \hat{OM'P'}, \quad \text{and} \quad \hat{MPO} = 90^\circ - A = \hat{M'OP'};$$

hence, by Euclid I. 26, the triangles MOP and $M'P'O$ are equal in all respects;

$$\therefore MP = OM', \quad \text{and} \quad OM = M'P', \quad \text{numerically};$$

$$\left. \begin{aligned} \therefore \sin(90^\circ + A) &= \sin XOB = \frac{M'P'}{OP'} = \frac{OM}{OP} = \cos A, \\ \cos(90^\circ + A) &= \cos XOB = \frac{OM'}{OP'} = \frac{-MP}{OP} = -\sin A, \\ \tan(90^\circ + A) &= \tan XOB = \frac{M'P'}{OM'} = \frac{OM}{-MP} = -\cot A; \end{aligned} \right\}$$

$$\left. \begin{aligned} \text{and similarly,} \quad \cot(90^\circ + A) &= -\tan A, \\ \sec(90^\circ + A) &= -\operatorname{cosec} A, \\ \operatorname{cosec}(90^\circ + A) &= \sec A. \end{aligned} \right\}$$

87. The results obtained in Arts. 85 and 86 may be stated briefly as follows:—In the case of each of the angles $90^\circ - A$, $90^\circ + A$, the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, *cosecant*, are respectively equal numerically to the *cosine*, *sine*, *cotangent*, *tangent*, *cosecant*, *secant*, of A ; i.e., any trigonometrical ratio of either of the angles $90^\circ - A$, $90^\circ + A$, is equal numerically to the corresponding co-ratio of A .

The *algebraic sign* of a ratio depends solely upon the position of the revolving line, and must be determined from it as in former cases. The student can, if necessary, refer to Art. 72.

EXAMPLES 30.

State the sine, cosine and tangent of

(1) $90^\circ - 45^\circ$.	(2) $90^\circ + 45^\circ$.	(3) $90^\circ - 30^\circ$.
(4) $90^\circ + 30^\circ$.	(5) $90^\circ - 60^\circ$.	(6) $90^\circ + 60^\circ$.
(7) $90^\circ + 15^\circ$.	(8) $90^\circ - 75^\circ$.	(9) $90^\circ + 75^\circ$.
(10) $\frac{\pi}{2} - \frac{\pi}{4}$.	(11) $\frac{\pi}{2} + \frac{\pi}{4}$.	(12) $\frac{\pi}{2} + \frac{\pi}{3}$.

Write down the value of

(13) $\cos 72^\circ$.	(14) $\cos 108^\circ$.	(15) $\sin 198^\circ$.

Apply the formulae for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$ to find the value of

(16) $\sin(180^\circ + A)$.	(17) $\sin(180^\circ - A)$.	(18) $\cos(180^\circ - A)$.
(19) $\cos(180^\circ + A)$.	(20) $\sin(90^\circ - A)$.	(21) $\cos(90^\circ - A)$.
(22) $\sin(90^\circ + A)$.	(23) $\cos(90^\circ + A)$.	(24) $\operatorname{cosec}(90^\circ - A)$.
(25) $\operatorname{cosec}(180^\circ - A)$.	(26) $\sec(90^\circ + A)$.	(27) $\tan(180^\circ + A)$.
(28) $\tan(180^\circ - A)$.	(29) $\cot(180^\circ - A)$.	(30) $\cot(180^\circ + A)$.

The changes in the value of a ratio as the angle increases from 0° to 360° .

88. We will now consider how the several trigonometrical ratios vary in magnitude and in sign as the revolving line, starting from its initial position OX , describes a complete revolution about O .

The matter is simplified by the understanding that OP remains always of the same length, and always positive in sign, to whatever position the revolving line may move. Consequently, to examine the manner in which the ratios vary in magnitude or sign, we have merely to consider in what manner the lengths MP and OM vary, *absolutely* for comparison with the constant length OP , and *relatively* for comparison with one another.

89. The variation of $\sin A$ as A increases from 0° to 360° .

Let XOP be the angle A . In OP take any point P , and draw PM meeting OX , or XO produced, at right angles at M .

(i) As A increases from 0° to 90° ,

MP increases from 0 to OP , and is positive,

$\therefore \frac{MP}{OP}$ increases numerically from 0 to 1, and is positive,

$\therefore \sin A$ increases numerically from 0 to 1, and is positive.

(ii) As A increases from 90° to 180° ,

MP decreases from OP to 0 , and is positive,

$\therefore \frac{MP}{OP}$ decreases numerically from 1 to 0 , and is positive,

$\therefore \sin A$ decreases numerically from 1 to 0 , and is positive.

(iii) As A increases from 180° to 270° ,

MP increases from 0 to OP , and is negative,

$\therefore \frac{MP}{OP}$ increases numerically from 0 to 1 , and is negative,

$\therefore \sin A$ increases numerically from 0 to 1 , and is negative.

(iv) As A increases from 270° to 360° ,

MP decreases from OP to 0 , and is negative,

$\therefore \frac{MP}{OP}$ decreases numerically from 1 to 0 , and is negative,

$\therefore \sin A$ decreases numerically from 1 to 0 , and is negative.

90. The variation of $\cos A$ as A increases from 0° to 360° .

(i) As A increases from 0° to 90° ,

OM decreases from OP to 0 , and is positive,

$\therefore \frac{OM}{OP}$ decreases numerically from 1 to 0 , and is positive,

$\therefore \cos A$ decreases numerically from 1 to 0 , and is positive.

- (ii) As A increases from 90° to 180° ,
 OM increases from 0 to OP , and is negative,
 $\therefore \frac{OM}{OP}$ increases numerically from 0 to 1 , and is negative,
 $\therefore \cos A$ increases numerically from 0 to 1 , and is negative.
- (iii) As A increases from 180° to 270° ,
 OM decreases from OP to 0 , and is negative,
 $\therefore \frac{OM}{OP}$ decreases numerically from 1 to 0 , and is negative,
 $\therefore \cos A$ decreases numerically from 1 to 0 , and is negative.
- (iv) As A increases from 270° to 360° ,
 OM increases from 0 to OP , and is positive,
 $\therefore \frac{OM}{OP}$ increases numerically from 0 to 1 , and is positive,
 $\therefore \cos A$ increases numerically from 0 to 1 , and is positive.

91. The variation of $\tan A$ as A increases from 0° to 360° .

- (i) As A increases from 0° to 90° ,
 MP increases from 0 to OP , and is positive, }
 and OM decreases from OP to 0 , and is positive, }
 $\therefore \frac{MP}{OM}$ increases numerically from 0 to ∞ , and is positive,
 $\therefore \tan A$ increases numerically from 0 to ∞ , and is positive.
- (ii) As A increases from 90° to 180° ,
 MP decreases from OP to 0 , and is positive, }
 and OM increases from 0 to OP , and is negative, }
 $\therefore \frac{MP}{OM}$ decreases numerically from ∞ to 0 , and is negative,
 $\therefore \tan A$ decreases numerically from ∞ to 0 , and is negative.
- (iii) As A increases from 180° to 270° ,
 MP increases from 0 to OP , and is negative, }
 OM decreases from OP to 0 , and is negative, }
 $\therefore \frac{MP}{OM}$ increases numerically from 0 to ∞ , and is positive,
 $\therefore \tan A$ increases numerically from 0 to ∞ , and is positive.
- (iv) As A increases from 270° to 360° ,
 MP decreases from OP to 0 , and is negative, }
 OM increases from 0 to OP , and is positive, }
 $\therefore \frac{MP}{OM}$ decreases numerically from ∞ to 0 , and is negative,
 $\therefore \tan A$ decreases numerically from ∞ to 0 , and is negative.

PART II.

LOGARITHMS AND MATHEMATICAL TABLES.

IX. LOGARITHMS.

92. It is known from algebra that for all values of the indices m and n , positive or negative, integral or fractional, the following three laws hold :—

$$x^m \times x^n = x^{m+n} \dots\dots\dots(i),$$

$$x^m \div x^n = x^{m-n} \dots\dots\dots(ii),$$

and

$$(x^m)^n = x^{mn} \dots\dots\dots(iii),$$

Thus the *product* of two powers of the same number is obtained by *adding* the indices of the factors, and the *quotient* when one power is divided by another is obtained by *subtracting* the index of the latter from the index of the former.

By actual multiplication we find that

$$2^3 = 8, \quad 2^4 = 16, \quad 2^5 = 32, \quad 2^6 = 64, \quad 2^7 = 128, \quad \&c., \quad \&c.,$$

and we see that the numbers 3, 4, 5, 6, 7, &c., are the *indices* of the powers to which 2 must be raised to produce the numbers 8, 16, 32, 64, 128, &c., respectively. These *indices* are **logarithms**; thus :—

the **index** 3 is the **logarithm** of 8 to the **base** 2 ;

the **index** 4 is the **logarithm** of 16 to the **base** 2 ;

the **index** 5 is the **logarithm** of 32 to the **base** 2, &c.

A logarithm is therefore merely an index ; it has the same properties, and obeys the same laws.

93. DEFINITION. The **logarithm** of a number, to a given **base**, is the **index** of the power to which the **base** must be raised to produce the number; i.e., if $a^x = b$, then x is called the logarithm of b to the base a .

Any number whatever may be chosen as *base*. It may be positive or negative, integral or fractional, rational or irrational.

94. The fact that 3 is the logarithm of 8 to the base 2 is commonly expressed in the form, $\log_2 8 = 3$, which denotes the same as $2^3 = 8$. Similarly we may write $\log_2 16 = 4$, $\log_2 32 = 5$, $\log_3 243 = 5$, &c. Generally, $\log_a x = y$ denotes the same thing as $a^y = x$.

The following examples will serve as illustrations:

Example i. Find the logarithm of 243 to the base 3.

Let $x = \log_3 243$;
then, by definition, $3^x = 243 = 3^5$,
 $\therefore x = 5$,
 $\therefore \log_3 243 = 5$.

Example ii. Find the logarithm of 1000 to the base .01.

(a) Let $x = \log_{.01} 1000$;
then $(.01)^x = 1000$,
 $\therefore (\frac{1}{100})^x = 1000$,
 $\therefore 10^{-2x} = 10^3$,
 $\therefore x = -\frac{3}{2}$,
 $\therefore \log_{.01} 1000 = -\frac{3}{2}$.

(b) Otherwise; $1000 = 10^3$
 $= (100)^{\frac{3}{2}}$
 $= (\frac{1}{100})^{-\frac{3}{2}}$
 $= (.01)^{-\frac{3}{2}}$;
 $\therefore \log_{.01} 1000 = -\frac{3}{2}$.

EXAMPLES 31.

Find the logarithm of

(1) 8 to the base 2.	(2) 9 to the base 3.
(3) $\sqrt{2}$8.	(4) 16.....8.
(5) 32.....4.	(6) 27.....81.
(7) 81.....27.	(8) 25.....625.
(9) 49.....343.	(10) 256..... $\frac{1}{2}$.
(11) 10......01.	(12) .01.....1000.
(13) 1000......001.	(14) 81..... $\sqrt[3]{3}$.

Find the base when

(15) $\log 36 = 2$.	(16) $\log 81 = 4$.	(17) $\log 3 = \frac{1}{2}$.
(18) $\log 8 = \frac{1}{3}$.	(19) $\log 81 = 1\frac{1}{3}$.	(20) $\log 25 = -\frac{1}{2}$.

95. The following theorems hold in all cases, whatever the base may be. They are very important.

(i) **The logarithm of 1 is always 0.**

Let the base be a , and let $\log_a 1 = x$.

Then, by definition, $a^x = 1$;

but $a^0 = 1$, for all values of a ,

$$\therefore x = 0,$$

$$\therefore \log_a 1 = 0, \text{ whatever } a \text{ may be.}$$

(ii) **The logarithm of the base itself is always 1.**

Let the base be a , and let $\log_a a = x$.

Then, by definition, $a^x = a$,

$$\therefore x = 1,$$

$$\therefore \log_a a = 1, \text{ whatever } a \text{ may be.}$$

(iii) **The logarithm of a product is equal to the sum of the logarithms of the factors.**

Let $\log_a m = x$, } Then, by definition, $m = a^x$, }
and $\log_a n = y$. } and $n = a^y$; }

$$\therefore mn = a^{x+y},$$

$$\therefore \log_a (mn) = x + y,$$

$$\therefore \log_a (mn) = \log_a m + \log_a n.$$

Similarly, $\log_a (mnp \dots) = \log_a m + \log_a n + \log_a p + \dots$

Example. $\log 105 = \log (3 \times 5 \times 7) = \log 3 + \log 5 + \log 7$.

(iv) **The logarithm of a quotient is equal to the logarithm of the dividend, minus the logarithm of the divisor.**

Let $\log_a m = x$, } Then, by definition, $m = a^x$, }
and $\log_a n = y$. } and $n = a^y$; }

$$\therefore \frac{m}{n} = \frac{a^x}{a^y} = a^{x-y},$$

$$\therefore \log_a \left(\frac{m}{n} \right) = x - y = \log_a m - \log_a n.$$

Example. $\log 2\frac{1}{7} = \log \frac{15}{7} = \log 15 - \log 7 = \log 3 + \log 5 - \log 7$.

(v) The logarithm of any power of a number is equal to the logarithm of the number, multiplied by the index of the power.

Let $\log_a m = x$. Then, by definition, $m = a^x$,

$$\therefore m^r = a^{rx},$$

$$\therefore \log_a (m^r) = rx = r \log_a m.$$

Example. $\log 32 = \log 2^5 = 5 \log 2$, whatever the base may be;
and $\log \sqrt[3]{625} = \log 625^{\frac{1}{3}} = \log (5^4)^{\frac{1}{3}} = \log 5^{\frac{4}{3}} = \frac{4}{3} \log 5$.

96. The following is an important proposition.

To prove that $\log_b a \times \log_a b = 1$.

Let $\log_b a = x$, } then, by definition, $b^x = a$, }
and $\log_a b = y$; } and $a^y = b$; }
 $\therefore (b^x)^y = a^y = b$, $\therefore b^{xy} = b$, $\therefore xy = 1$, $\therefore \log_b a \times \log_a b = 1$.

EXAMPLES 32.

Prove that, whatever the base may be,

$$(1) \log 15 = \log 3 + \log 5. \quad (2) \log 55 = \log 5 + \log 11.$$

$$(3) \log 2\frac{1}{3} = \log 7 - \log 3. \quad (4) \log 49 = 2 \log 7.$$

$$(5) \log 64 = 6 \log 2. \quad (6) \log 5 = \log 10 - \log 2.$$

Prove that

$$(7) \log_3 6 = 1 + \log_3 2. \quad (8) \log_5 45 = 1 + 2 \log_5 3.$$

$$(9) \log_2 36 = 2 + 2 \log_2 3. \quad (10) \log_2 72 = 3 + 2 \log_2 3.$$

$$(11) \log_2 3\frac{3}{4} = \log_2 3 + \log_2 5 - 2.$$

Express in terms of the logarithms of their simplest factors:—

$$(12) \log \frac{abc^2}{d^3}. \quad (13) \log \sin 2A. \quad (14) \log \left(\frac{ab^3}{c^4} \right)^{\frac{2}{3}}.$$

$$(15) \log \sqrt{216}. \quad (16) \log \frac{1}{5} \sqrt{441}. \quad (17) \log \sqrt[3]{\frac{2 \cdot 4 \cdot 3}{2 \cdot 5 \cdot 4}}.$$

Common Logarithms.

97. Logarithms to the base 10 are called **common** logarithms. The practical advantages of using 10 as the base result from the fact that 10 is also the radix of the scale of notation in

which numbers are commonly expressed. We will give, for reference, the common logarithms of the first 10 integers;

$\log 1 = \cdot 0000000$	$\log 6 = \cdot 7781513$
$\log 2 = \cdot 3010300$	$\log 7 = \cdot 8450980$
$\log 3 = \cdot 4771213$	$\log 8 = \cdot 9030900$
$\log 4 = \cdot 6020600$	$\log 9 = \cdot 9542425$
$\log 5 = \cdot 6989700$	$\log 10 = 1\cdot 0000000$

DEFINITION. The *integral* part of a logarithm is called its **characteristic**, and the *decimal* part its **mantissa**.

When a logarithm is a common logarithm, we will omit the subscript number 10, and write simply $\log x$ instead of $\log_{10} x$.

98. The characteristics of the common logarithms of numbers *greater than* unity have an important property.

(i) The value of x which satisfies the equation $10^x = m$, will gradually increase as m increases. Consequently, since $1 = 10^0$ and $10 = 10^1$, the logarithms of numbers between 1 and 10 will lie between 0 and 1; i.e., the common logarithm of a number consisting of **one** digit will be a **pure decimal**.

(ii) Since $10 = 10^1$, and $100 = 10^2$, the common logarithms of all numbers between 10 and 100 will lie between 1 and 2; hence the common logarithm of any number of **two** digits will be **1 + a decimal**.

(iii) Similarly, the common logarithm of any number of **three** digits will be **2 + a decimal**; that of a number of **four** digits will be **3 + a decimal**; and so on.

It will be seen from these cases that **the characteristic of the common logarithm of a number greater than unity is one less than the number of digits in the integral part of the number**. For example, the *characteristic* of $\log 5\cdot 3257$ is 0, that of $\log 53257$ is 4, that of $\log 532\cdot 57$ is 2, &c.

99. The common logarithms of numbers *less than* unity have a similar property.

(i) For any number between 1 and $\cdot 1$, i.e., between 10^0 and 10^{-1} , the logarithm lies between 0 and -1 . This logarithm is therefore of the form $-$ (a pure decimal). It is found convenient however to have the mantissa always positive, and it is therefore customary to express this logarithm as $-1 +$ a decimal, or in the form $\bar{1} +$ a decimal. For example,

$$\log(\cdot 5) = \log \frac{1}{2} = \log 1 - \log 2 = 0 - \cdot 30103 = -1 + \cdot 69897 = \bar{1}\cdot 69897.$$

The characteristic of $\log \cdot 5$ is therefore -1 , and the mantissa is $\cdot 6989700$.

(ii) Similarly, for a number which lies between $\cdot 1$ and $\cdot 01$, i.e., between 10^{-1} and 10^{-2} , the logarithm lies between -1 and -2 , and is therefore of the form $-(1 + \text{a decimal})$, or $\bar{2} + \text{a decimal}$.

$$\begin{aligned}\text{For example, } \log(\cdot 05) &= \log\left(\frac{1}{20}\right) = -\log 20 = -(1\cdot 3010300) \\ &= -2 + \cdot 6989700 \\ &= \bar{2}\cdot 6989700.\end{aligned}$$

(iii) Similarly, it will be found that $\log(\cdot 005) = \bar{3}\cdot 6989700$.

From the above examples we see that for a *pure decimal*, the first digit of which is not zero, the characteristic is -1 ; and that, if the decimal point be followed by one, two, three, &c. zeros, the characteristics are -2 , -3 , -4 , &c., respectively. Consequently, **for a pure decimal the characteristic is negative in sign, and is numerically equal to one more than the number of zeros which follow immediately after the decimal point.** For example, the characteristic of $\log \cdot 00004562$ is $\bar{5}$, and that of $\log \cdot 00000004562$ is $\bar{8}$.

100. We may now consider the advantages referred to in Art. 97 as resulting from the use of 10 as base.

The logarithm of a product being obtained by adding together the logarithms of the factors, it follows that if a number be *multiplied* by 10, 100, 1000,, the logarithms of the products will be obtained by *adding* 1, 2, 3,, respectively, to the original logarithm. Similarly, if a number be *divided* by 10, 100, 1000,, the logarithms of the quotients will be obtained by *subtracting* 1, 2, 3, ..., respectively, from the original logarithm. For example, $\log 3271 = 3\cdot 5146805$, hence

$$\begin{aligned}\log 32710 &= 4\cdot 5146805, & \log 3271000 &= 6\cdot 5146805, \\ \log 327\cdot 1 &= 2\cdot 5146805, & \log 32\cdot 71 &= 1\cdot 5146805, \\ \log 3\cdot 271 &= 0\cdot 5146805, & \log \cdot 3271 &= \bar{1}\cdot 5146805.\end{aligned}$$

Thus it will be seen that so long as we have the same significant figures 3271, the *mantissa* of the logarithm is unchanged throughout, and is independent of the position of the decimal point. When the position of the decimal point changes the *characteristic* alone changes. The advantages of using 10 as the base are therefore that (a) the *characteristic* of a logarithm can be determined at sight, and (b) the *mantissa* remains unaltered when we insert a decimal point in the number, or remove it from the number, or transfer it from any one position to any other.

101. In the following example the student will obtain a little practice in working with logarithms which have negative characteristics.

Ex. i. Multiply $\bar{5}\cdot6289012$ by 3.

$$\begin{array}{r} \bar{5}\cdot6289012 \\ 3 \\ \hline \bar{14}\cdot8867036 \end{array}$$

Ex. ii. Divide $\bar{5}\cdot6289012$ by 9.

$$\begin{aligned} \frac{1}{9} \text{ of } \bar{5}\cdot6289012 &= \frac{1}{9}(-9 + 4\cdot6289012) \\ &= -1 + \cdot5143223 \\ &= \bar{1}\cdot5143223. \end{aligned}$$

EXAMPLES 33.

Add together

(1) $\bar{3}\cdot8980462$, $\bar{5}\cdot6784920$, $\bar{7}\cdot9105204$.

(2) $\bar{4}\cdot8001528$, $\bar{3}\cdot7906305$, $1\cdot6780123$.

(3) $\bar{1}\cdot2085627$, $\bar{4}\cdot0012689$, $6\cdot1423007$.

Subtract

(4) $\bar{7}\cdot69014$ from $\bar{9}\cdot20365$.

(5) $\bar{5}\cdot82015$ from $6\cdot60157$.

(6) $\bar{3}\cdot62918$ from $\bar{1}\cdot66540$.

Multiply

(7) $\bar{3}\cdot2084$ by 3, 5, 10.

(8) $\bar{1}\cdot4106$ by 6, 7, 64.

(9) $\bar{6}\cdot9847$ by 4, 9, 48.

Divide, to four places,

(10) $\bar{6}\cdot6214$ by 7, 9, 11.

(11) $\bar{19}\cdot0852$ by 10, 16, 25.

102. The following examples illustrate the various properties of logarithms which have been established.

Example i. Given $\log 2$, find $\log 5$.

$$\log 5 = \log \frac{1}{2} = \log 10 - \log 2 = 1 - \cdot3010300 = \cdot6989700.$$

NOTE. To subtract a pure decimal from an integer, we diminish the integer by 1, and subtract each digit of the decimal from 9 except the last; this last we subtract from 10. The digits of the result can thus be written down beginning at the left-hand.

Ex. ii. Given $\log 2$, $\log 3$, and $\log 7$, find $\log \frac{6}{7}$.

$$\log \frac{6}{7} = \log 6 - \log 7$$

$$= \log 2 + \log 3 - \log 7$$

$$= \left\{ \begin{array}{l} \cdot3010300 \\ + \cdot4771213 \end{array} \right\} - \cdot8450980$$

$$= \left\{ \begin{array}{l} \cdot7781513 \\ - \cdot8450980 \end{array} \right.$$

$$= - \cdot0669467$$

$$= \bar{1}\cdot9330533.$$

Ex. iii. Given $\log 2$, find $\log \cdot00128$.

$$\log \cdot00128 = \log 128 - \log 100000$$

$$= \log 2^7 - 5$$

$$= 7 \log 2 - 5$$

$$= 7 \times \cdot3010300 - 5$$

$$= \bar{3}\cdot10721.$$

Example iv. Given $\log 13$ and $\log 19$, find $\log (247)^{\frac{2}{3}}$.

$$\begin{aligned}\log (247)^{\frac{2}{3}} &= \frac{2}{3} \log 247 = \frac{2}{3} (\log 13 + \log 19) \\ &= \frac{2}{3} \left(\begin{array}{r} 1.1139434 \\ + 1.2787536 \end{array} \right) \\ &= \frac{2}{3} \times 2.3926970 \\ &= \frac{1}{3} \times 4.7853940 \\ &= 1.5951313.\end{aligned}$$

103. When logarithms are given in a question, it is to be understood that no others may be used in that question, except the logarithms of 1 and the various powers of 10.

EXAMPLES 34.

Given $\log 2$ and $\log 3$, find the common logarithms of

(1) 5.	(2) 9.	(3) 300.	(4) 25.
(5) 216.	(6) 72.	(7) 144.	(8) 108.
(9) 1080.	(10) $\frac{1}{100}$.	(11) .0002.	(12) .0048.
(13) .00096.	(14) $4\frac{1}{2}$.	(15) $2\frac{2}{5}$.	(16) .0000018.
(17) $(\frac{1}{256})^3$.	(18) .1875.	(19) $\cos 45^\circ$.	(20) $\sec 30^\circ$.
(21) $\sqrt[3]{\frac{1}{6}}$.	(22) $(8\frac{1}{3})^{\frac{1}{5}}$.	(23) $\sqrt[3]{.0012}$.	(24) $(.0025)^{\frac{1}{4}}$.

104. The following example will further illustrate the connection between the number of digits in the integral part of a number and the characteristic of its logarithm.

Example. Given $\log 2$, find how many digits there are in the integral part of $(1.28)^{5000}$.

$$\begin{aligned}\log (1.28)^{5000} &= 5000 \log 1.28 = 5000 (\log 2^7 - \log 10^2) \\ &= 5000 (7 \log 2 - 2 \log 10) = 5000 (7 \times .30103 - 2) \\ &= 5000 \times .10721 \\ &= 536.05 ;\end{aligned}$$

$\therefore$ no. of digits in the integral part of $(1.28)^{5000}$ is 537 (Art. 98).

EXAMPLES 35.

What are the characteristics of the common logarithms of

(1) 327.	(2) 3274.	(3) 32748.	(4) 327482.
(5) 32748295.	(6) .327.	(7) .03274.	(8) .00327.
(9) .000327.	(10) .00000000589.	(11) .00000000000005.	
(12) $\frac{1}{56}$.	(13) $\frac{1}{482}$.	(14) $\frac{41}{763}$.	(15) $\frac{34}{89061}$?

Find the number of digits in the integral part of

- (16) $(4.9)^{300}$; using $\log 7$. (17) $(25.6)^{250}$; using $\log 2$.
 (18) $(1.1)^{8000}$; using $\log 11 = 1.0413927$.
 (19) $(1.04)^{4000}$; using $\log 104 = 2.0170333$.
 (20) $(5.15)^{5000}$; using $\log 2$, and $\log 103 = 2.0128372$.

Find how many zeros there are before the first significant figure in the following numbers when expressed as decimals; using the logarithms given on page 71.

- (21) $(\frac{1}{2})^{200}$. (22) $(\frac{1}{3})^{80}$. (23) $(\frac{2}{5})^{150}$. (24) $(\frac{3}{8})^{40}$. (25) $(\frac{1}{7})^{5000}$.

EXAMPLES 36. MISCELLANEOUS.

(1) What is the number whose logarithm is the same to every base?

(2) What is the logarithm to the base 3 of (i) $\sqrt{3}$, (ii) $\frac{1}{\sqrt{3}}$, (iii) 81?

(3) What is the characteristic of $\log_4 27$, $\log_3 .004$ and $\log_6 1735$?

(4) Within what limits does a number lie if its common logarithm has for characteristic (i) 3, (ii) 0, (iii) -4 ?

(5) Find the numbers whose logarithms to the base 64 are $\frac{1}{3}$, $1\frac{1}{2}$, 2, $1\frac{1}{3}$, and $1\frac{1}{6}$.

(6) Given $\log 105 = 2.0211893$, find $\log .0000105$.

(7) Given $\log 2$, $\log 3$ and $\log 7$, find the common logarithms of (i) 42, (ii) $.84$, (iii) $.003$, (iv) $\sqrt{15}$, (v) $(18)^{-\frac{1}{2}}$, (vi) $(24.5)^{-\frac{1}{3}}$.

(8) Given $\log 2$ and $\log 3$, find how many digits there are in 2^{50} , 12^{20} and 15^{100} , and how many zeros before the first significant figure in $(\frac{2}{3})^{25}$.

(9) Given $\log 2$ and $\log 3$, find the number of digits in the integral part of $(8\frac{1}{3})^9$.

(10) Given $\log 1.03 = .0128372$, determine whether $(1.03)^{80}$ is greater than, or less than, 10.

(11) Find $\log .064$ in terms of $\log 5$.

(12) Given $\log 5$, find the logarithms of (i) 2, (ii) $\left(\frac{2^{60}}{5^{20}}\right)^{\frac{1}{14}}$.

(13) Given $\log 25 = 1.3979400$, find the logarithms of 5, 2 and $(.064)^{\frac{1}{10}}$.

(14) Given $\log 2$, $\log 3$, and $\log 3301.93 = 3.5187675$, find the value of $\sqrt[3]{36}$.

(15) Given $\log 24 = 1.380211$, and $\log 36 = 1.556303$, find $\log 54$, and $\log\left(\frac{3}{5}\right)$.

(16) Find the value of $7 \log_2 \left(\frac{1}{15}\right) + 5 \log_2 \left(\frac{25}{4}\right) + 3 \log_2 \left(\frac{81}{80}\right)$.

X. THE USE OF MATHEMATICAL TABLES.

105. DEFINITION. The numerical values of the sine, cosine, tangent, &c., of an angle are called respectively the **natural sine**, **natural cosine**, **natural tangent**, &c. The logarithms of these numerical values are called respectively the **logarithmic sine**, **logarithmic cosine**, **logarithmic tangent**, &c.

We know that for all angles the sines and the cosines, and in many cases other ratios also, are numerically less than unity, and consequently that the logarithms of such ratios are negative. To avoid the inconvenience of printing tables of negative logarithms it is customary to increase each logarithmic ratio by 10. The results are called **Tabular logarithms**, and it is these that are recorded in books of Mathematical Tables.

106. DEFINITION. A **Tabular logarithmic ratio** is the common logarithmic ratio increased by 10.

Logarithms of Numbers.

107. It is only in a comparatively few cases such as 1, 10, $10^2, \dots, 10^{-1}, 10^{-2}, \dots$, &c., that a common logarithm is an integer; and only in a few, such as $10^{\frac{1}{2}}, 10^{\frac{1}{4}}, 10^{\frac{1}{8}}, \dots$, &c., is it a terminating decimal. In the great majority of cases a logarithm is a decimal which neither terminates nor recurs. The mantissæ are usually given in the Tables as far as the 7th decimal place; but no characteristics are recorded there, for these can be ascertained by inspection.

108. To find the mantissa belonging to a number intermediate between two consecutive numbers whose mantissæ are given in the tables, we adopt a method which is based upon a principle which holds good when we are concerned with only 7 places of decimals, namely, that **when the change in a number is small, the change in the number and the change in the logarithm are proportional to one another.** This is called the Principle of **Proportional Parts**.

Whatever a number may be whose logarithm is given, the student should always begin by placing a decimal point immediately after the first significant figure, as in Example ii, so that the logarithm may consist of a mantissa only.

Example i. Given $\log 5.7826 = .7621232$, and $\log 5.7827 = .7621307$; determine $\log 5.782643$.

We have $\log 5.7826 = .7621232$,

and $\log 5.7827 = .7621307$,

$\therefore$ difference for $.0001 = .0000075$;

$\therefore$ difference for $.000043 = \frac{43}{1000}$ of $.0000075 = .000003225$;

but $\log 5.782600 = .7621232$;

$\therefore$ by addition, $\log 5.782643 = .7621264$.

Ex. ii. Given $\log .0032853 = \bar{3}.5165750$, and $\log 328.54 = 2.5165883$, find $\log .000032853249$.

We have $\log 3.2853 = .5165750$,

and $\log 3.2854 = .5165883$;

$\therefore$ diff. for $.0001 = .0000133$.

In making the calculation it will be found convenient to state the work thus :—

$$\begin{array}{rcl} \log 3.2853 & = & .5165750 \\ & 2 & 266 \\ & 4 & 532 \\ & 9 & 1197 \\ \log 3.2853249 & = & .5165783117 \\ & & = .5165783 \text{ to seven places} \end{array}$$

$\therefore \log .000032853249 = \bar{5}.5165783$.

NOTE. If the digit in the 8th place of this logarithm had been 5, 6, 7, 8 or 9 we would, in retaining 7 places only, have changed the digit in the 7th place from 3 to 4 (*Arithmetic*, Art. 203 b).

EXAMPLES 37.

- (1) Given $\log 6.8091 = .8330897$
 and $\log 6.8092 = .8330961$ } find $\log 6.80915$.
- (2) Given $\log 2.1004 = .3223020$
 and $\log 2.1005 = .3223227$ } find $\log 2.100483$.
- (3) Given $\log 43.297 = 1.6364578$
 and $\log 432.98 = 2.6364678$ } find $\log .0004329747$.
- (4) Given $\log 863.01 = 2.9360158$
 and $\log .00086302 = \bar{4}.9360209$ } find $\log 86301753$.

109. The converse problem, to determine the number to which a given logarithm belongs, depends upon the same property, that to a small change in the logarithm corresponds a *proportional* change in the number.

Example. Given $\log 3.7801 = .5775033$, and $\log 3.7802 = .5775148$; find the number of which $\bar{3}.5775120$ is the logarithm.

Let the required number be $3.7801 + x$.

We have

$$\log 3.7801 = .5775033,$$

$$\text{diff. for } .0001 = .0000115,$$

$$\text{and diff. for } x = .0000087,$$

$$\therefore x = \frac{.87}{115} \times .0001 = .00007565,$$

$$\therefore \text{by addition, } \log 3.78017565 = .5775120,$$

$$\therefore \log .00378017565 = \bar{3}.5775120,$$

$$\therefore \text{number required} = .00378017565.$$

EXAMPLES 38.

Determine the number whose logarithm is

- (1) $.6870891$; given $\log 4.8650 = .6870828$
 and $\log 4.8651 = .6870918$.
- (2) $.9283642$; given $\log 8.4793 = .9283600$
 and $\log 8.4794 = .9283651$.
- (3) 1.8469892 ; given $\log 7.0305 = .8469862$
 and $\log 7.0306 = .8469924$.
- (4) $\bar{2}.3822111$; given $\log 2.4110 = .3821972$
 and $\log 2.4111 = .3822152$.

Natural Ratios.

110. The tables give the natural ratios of all angles in the first quadrant at angular intervals of one minute.

To determine a ratio for an angle *intermediate* between two of those whose ratios are recorded, we proceed in just the same way as when we find the logarithm of a number intermediate between two consecutive tabulated numbers. The same property holds good here also; namely, that, as far as 7 decimal places, for a small change in an angle there is, except in certain special cases, a *proportional* change in any ratio belonging to it.

Example. Given $\sin 35^\circ 26' = \cdot 5797553$, and $\sin 35^\circ 27' = \cdot 5799923$; find $\sin 35^\circ 26' 42''$.

We have $\sin 35^\circ 26' = \cdot 5797553$,

and difference for $60'' = \cdot 0002370$;

$\therefore$ difference for $42'' = \frac{42}{60} \times \cdot 0002370 = \cdot 0001659$;

$\therefore$ by *adding* to $\sin 35^\circ 26'$ we get

$\sin 35^\circ 26' 42'' = \cdot 5799212$.

EXAMPLES 39.

$$\left. \begin{array}{l} (1) \text{ Given } \sin 23^\circ 15' = \cdot 3947439 \\ \sin 23^\circ 16' = \cdot 3950111 \end{array} \right\} \text{ find } \sin 23^\circ 15' 20''.$$

$$\left. \begin{array}{l} (2) \text{ Given } \sin 48^\circ 2' = \cdot 7435340 \\ \sin 48^\circ 3' = \cdot 7437285 \end{array} \right\} \text{ find } \sin 48^\circ 2' 32''.$$

$$\left. \begin{array}{l} (3) \text{ Given } \tan 32^\circ 17' = \cdot 6317667 \\ \tan 32^\circ 18' = \cdot 6321738 \end{array} \right\} \text{ find } \tan 32^\circ 17' 40''.$$

$$\left. \begin{array}{l} (4) \text{ Given } \tan 28^\circ 41' = \cdot 5471060 \\ \tan 28^\circ 42' = \cdot 5474840 \end{array} \right\} \text{ find } \tan 28^\circ 41' 25'' \cdot 5.$$

$$\left. \begin{array}{l} (5) \text{ Given } \sec 42^\circ 18' = 1 \cdot 3520254 \\ \sec 42^\circ 19' = 1 \cdot 3523834 \end{array} \right\} \text{ find } \sec 42^\circ 18' 35'' \cdot 4.$$

111. The converse problem, to determine the angle to which a given sine, tangent, or secant belongs, is similar to the corresponding problem with respect to numbers, and is solved in a manner similar to that of Art. 109.

Example. Given $\tan 73^\circ 39' = 3.4086882$, and $\tan 73^\circ 40' = 3.4123626$; find the angle whose tangent is 3.4102848 .

Let the required angle be $73^\circ 39' x''$.

We have $\tan 73^\circ 39' = 3.4086882$.

Difference for $60'' = .0036744$,

and difference for $x'' = .0015966$;

$$\therefore x = 60 \times \frac{.0015966}{.0036744} = 26.07;$$

$\therefore$ the required angle $= 73^\circ 39' 26''.07$.

EXAMPLES 40.

- (1) Given $\sin 54^\circ 36' = .8151278$ } determine the angle whose
 $\sin 54^\circ 37' = .8152963$ } sine is $.8152346$.
- (2) Given $\tan 16^\circ 17' = .2921047$ } determine the angle whose
 $\tan 16^\circ 18' = .2924205$ } tangent is $.2922156$.
- (3) Given $\tan 62^\circ 23' = 1.9114691$ } determine the angle whose
 $\tan 62^\circ 24' = 1.9128236$ } tangent is 1.9125129 .
- (4) Given $\sec 58^\circ 2' = 1.8888388$ } determine the angle whose
 $\sec 58^\circ 3' = 1.8897197$ } secant is 1.8892105 .
- (5) Given $\sec 26^\circ 45' = 1.1198472$ } determine the angle whose
 $\sec 26^\circ 46' = 1.1200115$ } secant is 1.1199363 .

112. For angles in the first quadrant, the cosine, cotangent and cosecant *decrease* as the angle increases. Consequently the calculated ratio-difference for a small angle-difference must in these cases be *subtracted*, as in the example which follows.

Example. Given $\cos 23^\circ 41' = .9157795$, and $\cos 23^\circ 42' = .9156626$; determine the cosine of $23^\circ 41' 37''$.

We have $\cos 23^\circ 41' = .9157795$.

Difference for $60'' = .0001169$;

$$\therefore \text{difference for } 37'' = \frac{37}{60} \text{ of } .0001169 = .0000721;$$

$$\therefore \cos 23^\circ 41' 37'' = \begin{cases} .9157795 \\ - .0000721 \end{cases} = .9157074.$$

Conversely, to determine the angle which has a given cosine, cotangent or cosecant, we proceed as in Art. 111, remembering only that the angle *decreases* as these three ratios increase.

Example. Given $\cot 27^\circ 5' = 1.9555739$, and $\cot 27^\circ 6' = 1.9541713$; determine the angle whose cotangent is 1.9552618 .

Let the required angle be $27^\circ 5' x''$;

then diff. for $60'' = 1.9555739 - 1.9541713 = .0014026$,

and diff. for $x'' = 1.9555739 - 1.9552618 = .0003121$;

$$\therefore x = 60 \times \frac{.0003121}{.0014026} = 13.35;$$

$\therefore$ angle required $= 27^\circ 5' 13''.35$.

EXAMPLES 41.

(1) Given $\cos 9^\circ 28' = .9863815$
and $\cos 9^\circ 29' = .9863336$

find (i) $\cos 9^\circ 28' 15''$, (ii) the angle whose cosine is $.9863518$.

(2) Given $\cos 31^\circ 19' = .8543077$
and $\cos 31^\circ 20' = .8541564$

find (i) $\cos 31^\circ 19' 33''$, (ii) the angle whose cosine is $.8542563$.

(3) Given $\cot 81^\circ 46' = .1446961$
and $\cot 81^\circ 47' = .1443991$

find (i) $\cot 81^\circ 46' 51''$, (ii) the angle whose cot. is $.1445283$.

(4) Given $\cot 63^\circ 24' = .5007627$
and $\cot 63^\circ 25' = .5003989$

find (i) $\cot 63^\circ 24' 43''$, (ii) the angle whose cot. is $.5005263$.

(5) Given $\operatorname{cosec} 57^\circ 33' = 1.1850307$
and $\operatorname{cosec} 57^\circ 34' = 1.1848116$

find (i) $\operatorname{cosec} 57^\circ 33' 39''.5$, (ii) the angle whose cosec. is 1.1849208 .

Logarithmic Ratios.

113. We have said that the logarithmic ratios recorded in the Tables are the common logarithms increased by 10. For example, the tabular logarithm of $\cos 60^\circ$

$$= \log \cos 60^\circ + 10 = \log \frac{1}{2} + 10 = -.30103 + 10 = 9.69897.$$

A tabular logarithm is distinguished from a common logarithm by the symbol L ; thus $\log \cos 60^\circ = -.30103$,

$$\text{and } L \cos 60^\circ = 9.69897.$$

Just as in the case of numbers, the principle of **proportional parts** may be applied to the calculation of the *logarithmic ratios* of intermediate angles, and also to the converse problem, the determination of the angle to which a given logarithmic ratio belongs. The principle is not universally applicable. It fails in certain cases when the angle is very small, and when it is very nearly a right angle, but such cases will not occur in the examples which will here be put before the student.

EXAMPLES 42.

$$(1) \quad \text{Given} \quad \begin{array}{l} L \sin 24^\circ 15' = 9.6135446 \\ L \sin 24^\circ 16' = 9.6138250 \end{array}$$

find (i) $L \sin 24^\circ 15' 46''$, (ii) the angle whose L sine is 9.6137031.

$$(2) \quad \text{Given} \quad \begin{array}{l} L \tan 67^\circ 29' = 10.3824185 \\ L \tan 67^\circ 30' = 10.3827757 \end{array}$$

find (i) $L \tan 67^\circ 29' 52''$, (ii) the angle whose L tan. is 10.3825849.

$$(3) \quad \text{Given} \quad \begin{array}{l} L \sec 54^\circ 48' = 10.2392517 \\ L \sec 54^\circ 49' = 10.2394308 \end{array}$$

find (i) $L \sec 54^\circ 48' 31''$, (ii) the angle whose L sec. is 10.2393106.

$$(4) \quad \text{Given} \quad \begin{array}{l} L \cos 74^\circ 41' = 9.4218566 \\ L \cos 74^\circ 42' = 9.4213950 \end{array}$$

find (i) $L \cos 74^\circ 41' 13''$, (ii) the angle whose L cos. is 9.4217415.

$$(5) \quad \text{Given} \quad \begin{array}{l} L \cot 63^\circ 27' = 9.6986847 \\ L \cot 63^\circ 28' = 9.6983687 \end{array}$$

find (i) $L \cot 63^\circ 27' 24''$, (ii) the angle whose L cot. is 9.6984763.

$$(6) \quad \text{Given} \quad \begin{array}{l} L \operatorname{cosec} 37^\circ 21' = 10.2170386 \\ L \operatorname{cosec} 37^\circ 22' = 10.2168732 \end{array}$$

find (i) $L \operatorname{cosec} 37^\circ 21' 53''$,

(ii) the angle whose L cosec is 10.2169452.

114. We will close this section with an example of the use of logarithms in determining any root of a number.

Example. Determine the value of $\sqrt[7]{3625\cdot4}$.

$$\log \sqrt[7]{3625\cdot4} = \frac{1}{7} \log 3625\cdot4 = \frac{1}{7} (3\cdot5593559) = \cdot5084794.$$

$$\begin{array}{l} \text{But } \log 3\cdot2247 = \cdot5084893 \\ \text{and } \log 3\cdot2246 = \cdot5084758 \end{array} \Bigg\} ;$$

$$\therefore \text{ difference for } \cdot0001 = \cdot0000135,$$

$$\text{and } \log \text{ of reqd no.} - \log 3\cdot2246 = \cdot0000036 ;$$

$$\therefore \text{ required difference in number} = \frac{3\cdot6}{1\cdot35} \text{ of } \cdot0001 = \cdot000026...;$$

$$\therefore \text{ required number} = 3\cdot224626...;$$

$$\therefore \sqrt[7]{3625\cdot4} = 3\cdot224626...$$

115. If $L \tan \theta$ be given, $L \cot \theta$ may be obtained as follows :—

$$L \cot \theta = 10 + \log \cot \theta = 10 + \log \frac{1}{\tan \theta} = 10 - \log \tan \theta = 20 - L \tan \theta.$$

Similarly, $L \operatorname{cosec} \theta = 20 - L \sin \theta$, and $L \sec \theta = 20 - L \cos \theta$.

EXAMPLES 43.

(1) Given $\log 2$ and $\log 3$, find $\log 3\frac{6}{25}$.

(2) Given $\log 2$, find $\log \cdot250$, $L \sin 30^\circ$ and $L \operatorname{cosec} 45^\circ$.

(3) Given $\log 5 = \cdot69897$, find $\log 200$, $\log \sqrt[3]{62\cdot5}$, $L \cos 60^\circ$, and $L \cos 45^\circ$.

(4) Find $\log_{3\sqrt{11}} (297\sqrt{11})^{\frac{3}{5}}$.

(5) Given $\log 4\cdot96 = \cdot6954817$
and $\log 4\cdot9601 = \cdot6954904$

find the logarithms of 496010, $\cdot000496$ and $49600\cdot25$.

(6) Given $\log 3\cdot674 = \cdot5651392$
and $\log 3\cdot675 = \cdot5651510$

calculate the logarithms of $\cdot003674$, 36741 and $\sqrt{3674 \times 3675}$.

(7) Given $\log 3\cdot1415 = \cdot4971371$, and difference = $\cdot0000139$, find how many zeros follow the decimal point in the value of $(\cdot000314159)^{10}$.

(8) Given $\log 97941 = 4.9909645$, and $\log 63864 = 4.8052561$, find the logarithm of $97.941 \times .0063864$.

(9) Find the value of $(1.01)^{27}$, given $\log 1.01 = .00432$ and $\log 13081 = 4.11664$.

(10) Find the value of $\sqrt[7]{\frac{13}{300}}$; given

$\log 3$, $\log 13 = 1.11394$ and $\log 63865 = 4.80526$.

(11) Find the value of $\sqrt[7]{.000026751}$; given

$\log 26751 = 4.42734$ and $\log 22221 = 4.34676$.

(12) Given $\log 3.6412 = .5612445$,

and difference for $.0001 = .0000120$; also $\log 60020 = 4.7782958$; find the 11th root of $.003641279$ to 5 decimal places.

(13) Given $\log 1.384 = .1411361$, $\log 1.015 = .0064660$, and $\log 1.02985 = .0127726$, calculate the 7th root of $1.384 \div (1.015)^8$.

(14) If $9^x \div 3^{x+y} = 27$, and $2x = 5y$, find x and y .

(15) If $L \sin 15^\circ = 9.4129962$, find the value of $L \cos 15^\circ$. (Art. 59.)

(16) Given $\log 2$, $\log 3$, and $\log 7$, and $\log 3.7689 = .576214$, shew that

$$64 \left\{ 1 - \frac{1}{(1.05)^{20}} \right\} = 39.879 \text{ nearly.}$$

PART III.

TRIANGLES.

XI. THE ANGLES OF A TRIANGLE.

116. To denote the sides and angles of a triangle ABC , the following symbols are used:—

A , B , and C for the angles BAC , ABC , and ACB respectively; a , b , c for the number of units of length in the sides opposite A , B and C respectively; $2s$ for the *perimeter*, i.e., for $a + b + c$.

The unit of length may be an inch, or a mile, or any length we please, but of course it must be the same for all lengths throughout any investigation.

The six quantities A , B , C , a , b , c are called the **elements** of the triangle.

117. If A , B and C be the angles of a triangle, we know that $A + B + C = 180^\circ$. The relations which exist between the trigonometrical ratios of three angles which satisfy this condition are of great importance.

Example. If $A + B + C = 180^\circ$, prove that $\cos(A + B) = -\cos C$.

$$\begin{aligned}\cos(A + B) &= \cos(180^\circ - C) \\ &= -\cos C.\end{aligned}$$

EXAMPLES 44.

If $A + B + C = 180^\circ$, prove that

$$(1) \quad \sin(A + B) = \sin C. \quad (2) \quad \tan(A + B) = -\tan C.$$

$$(3) \quad \cot(A + B) = -\cot C. \quad (4) \quad \sec(A + B) = -\sec C.$$

$$(5) \quad \sin \frac{1}{2}(A + B) = \cos \frac{C}{2}. \quad (6) \quad \cos \frac{1}{2}(A + B) = \sin \frac{C}{2}.$$

$$(7) \quad \tan \frac{1}{2}(A + B) = \cot \frac{C}{2}. \quad (8) \quad \cot \frac{1}{2}(A + B) = \tan \frac{C}{2}.$$

$$(9) \quad \operatorname{cosec} \frac{1}{2}(A + B) = \sec \frac{C}{2}. \quad (10) \quad \sec \frac{1}{2}(A + B) = \operatorname{cosec} \frac{C}{2}.$$

118. The following relations, which hold when $A + B + C = 180^\circ$, are important both in themselves, and also as exemplifying the use of the various formulæ which have been established in the preceding pages;—

$$(i) \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C;$$

$$(ii) \quad \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1;$$

$$(iii) \quad \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C;$$

$$(iv) \quad \sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2};$$

$$(v) \quad \cos A + \cos B + \cos C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1.$$

PROOFS.

$$(i) \quad A + B + C = 180^\circ,$$

$$\therefore A + B = 180^\circ - C,$$

$$\therefore \tan(A + B) = -\tan C,$$

$$\therefore \frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C,$$

$$\therefore \tan A + \tan B = -\tan C + \tan A \tan B \tan C,$$

$$\therefore \tan A + \tan B + \tan C = \tan A \tan B \tan C.$$

Or, multiplying throughout by $\cot A \cot B \cot C$, we get

$$\cot B \cot C + \cot C \cot A + \cot A \cot B = 1.$$

$$\begin{aligned}
 \text{(ii)} \quad & A+B+C=180^\circ, \\
 & \therefore \frac{A}{2} + \frac{B}{2} = 90^\circ - \frac{C}{2}, \\
 & \therefore \tan \left(\frac{A}{2} + \frac{B}{2} \right) = \cot \frac{C}{2},
 \end{aligned}$$

Proceeding as in (i) we find that

$$\tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} + \tan \frac{A}{2} \tan \frac{B}{2} = 1.$$

$$\begin{aligned}
 \text{(iii)} \quad \sin 2A + \sin 2B + \sin 2C &= 2 \sin (A+B) \cos (A-B) + \sin 2C \\
 &= 2 \sin C \cos (A-B) + 2 \sin C \cos C \\
 &= 2 \sin C \{ \cos (A-B) + \cos C \} \\
 &= 2 \sin C \{ \cos (A-B) - \cos (A+B) \} \\
 &= 2 \sin C (2 \sin A \sin B) \\
 &= 4 \sin A \sin B \sin C.
 \end{aligned}$$

$$\begin{aligned}
 \text{(iv)} \quad \sin A + \sin B + \sin C &= 2 \sin \frac{A+B}{2} \cos \frac{A-B}{2} + \sin C \\
 &= 2 \cos \frac{C}{2} \cos \frac{A-B}{2} + 2 \sin \frac{C}{2} \cos \frac{C}{2} \\
 &= 2 \cos \frac{C}{2} \left\{ \cos \frac{A-B}{2} + \sin \frac{C}{2} \right\} \\
 &= 2 \cos \frac{C}{2} \left\{ \cos \frac{A-B}{2} + \cos \frac{A+B}{2} \right\} \\
 &= 2 \cos \frac{C}{2} \left(2 \cos \frac{A}{2} \cos \frac{B}{2} \right) \\
 &= 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.
 \end{aligned}$$

$$\begin{aligned}
 \text{(v)} \quad \cos A + \cos B + \cos C &= 2 \cos \frac{A+B}{2} \cos \frac{A-B}{2} + \cos C \\
 &= 2 \sin \frac{C}{2} \cos \frac{A-B}{2} + 1 - 2 \sin^2 \frac{C}{2} \\
 &= 2 \sin \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \sin \frac{C}{2} \right\} + 1 \\
 &= 2 \sin \frac{C}{2} \left\{ \cos \frac{A-B}{2} - \cos \frac{A+B}{2} \right\} + 1 \\
 &= 2 \sin \frac{C}{2} \left(2 \sin \frac{A}{2} \sin \frac{B}{2} \right) + 1 \\
 &= 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1
 \end{aligned}$$

EXAMPLES 45.

Prove that

$$(1) \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C,$$

when (i) $A + B + C = 0$, (ii) $A + B + C = 360^\circ$.

$$(2) \quad \tan 3A - \tan 2A - \tan A = \tan 3A \tan 2A \tan A.$$

$$(3) \quad \tan 5A - \tan 2A - \tan 3A = \tan 2A \tan 3A \tan 5A.$$

$$(4) \quad \tan 24^\circ + \tan 75^\circ + \tan 81^\circ = \tan 24^\circ \tan 75^\circ \tan 81^\circ.$$

If A , B and C be the angles of a triangle, prove that

$$(5) \quad \sin 2A - \sin 2B + \sin 2C = 4 \cos A \sin B \cos C.$$

$$(6) \quad \sin 2A + \sin 2B - \sin 2C = 4 \cos A \cos B \sin C.$$

$$(7) \quad \cos 2A + \cos 2B + \cos 2C = -1 - 4 \cos A \cos B \cos C.$$

$$(8) \quad \cos 2A - \cos 2B + \cos 2C = 1 - 4 \sin A \cos B \sin C.$$

$$(9) \quad \cos 2A + \cos 2B - \cos 2C = 1 - 4 \sin A \sin B \cos C.$$

$$(10) \quad \sin A + \sin B - \sin C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$$

$$(11) \quad \sin A - \sin B + \sin C = 4 \sin \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

$$(12) \quad \cos A + \cos B - \cos C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2} - 1.$$

$$(13) \quad \cos A - \cos B + \cos C = 4 \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2} - 1.$$

$$(14) \quad \sin B + \sin C - \sin A = 4 \cos \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$(15) \quad \cos B + \cos C - \cos A = 4 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

$$(16) \quad \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1 - 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

$$(17) \quad \cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} = 2 + 2 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

119. The following example illustrates the converse operation, the expression of a product of sines and cosines as an algebraic sum.

Example. If $A+B+C=180^\circ$, prove that

$$4 \sin A \cos B \sin C = 1 - \cos 2A + \cos 2B - \cos 2C.$$

$$\begin{aligned} 4 \sin A \cos B \sin C &= 2 \sin A (2 \cos B \sin C) \\ &= 2 \sin A \{\sin (B+C) - \sin (B-C)\} \\ &= 2 \sin A \{\sin A - \sin (B-C)\} \\ &= 2 \sin^2 A - 2 \sin A \sin (B-C) \\ &= 2 \sin^2 A - 2 \sin (B+C) \sin (B-C) \\ &= 1 - \cos 2A - (\cos 2C - \cos 2B) \\ &= 1 - \cos 2A + \cos 2B - \cos 2C. \end{aligned}$$

EXAMPLES 46.

Convert each of the following expressions into an algebraic sum, assuming that $A+B+C=180^\circ$.

(1) $4 \sin A \sin B \sin C.$	(2) $4 \sin A \sin B \cos C.$
(3) $4 \sin A \cos B \cos C.$	(4) $4 \cos A \cos B \cos C.$
(5) $4 \cos A \sin B \cos C.$	(6) $4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$
(7) $4 \sin \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2}.$	(8) $4 \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$
(9) $4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$	(10) $4 \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$

XII. THE SIDES AND ANGLES OF A TRIANGLE.

(i) A right-angled triangle.

120. In a triangle having a right angle at C we can at once from the figure write down any trigonometrical ratio of A or B in terms of the sides. Thus

$$\sin A = \frac{a}{c}, \quad \cos A = \frac{b}{c}, \quad \tan A = \frac{a}{b},$$

$$\cot A = \frac{b}{a}, \quad \sec A = \frac{c}{b}, \quad \operatorname{cosec} A = \frac{c}{a},$$

$$\sin B = \frac{b}{c}, \quad \cos B = \frac{a}{c}, \quad \tan B = \frac{b}{a},$$

$$\cot B = \frac{a}{b}, \quad \sec B = \frac{c}{a}, \quad \operatorname{cosec} B = \frac{c}{b}.$$

These relations may also be written thus:—

$$\left. \begin{aligned} a &= c \sin A = b \tan A &= c \cos B = b \cot B; \\ b &= c \cos A = a \cot A &= c \sin B = a \tan B; \\ c &= b \sec A = a \operatorname{cosec} A = a \sec B = b \operatorname{cosec} B. \end{aligned} \right\}$$

From these relations we may at once deduce that $c^2 = a^2 + b^2$, a formula which agrees with Euclid I. 47.

121. A few examples involving the relations given in the preceding article will enable the student to make himself familiar with their use. We will work out one example for his guidance.

Example. Express $\frac{\sin(A-B)}{\sin C}$ in terms of the sides.

$$\begin{aligned} \frac{\sin(A-B)}{\sin C} &= \frac{\sin A \cos B - \cos A \sin B}{\sin C} \\ &= \sin A \cos B - \cos A \sin B \\ &= \frac{a}{c} \times \frac{a}{c} - \frac{b}{c} \times \frac{b}{c} \\ &= \frac{a^2 - b^2}{c^2}. \end{aligned}$$

EXAMPLES 47.

Prove, in the case of a triangle right-angled at C, that

- (1) $b \cos A \cot B + c \cot C = \frac{ab}{c}.$
- (2) $b(\cos A + \sin A \cot B) = c.$
- (3) $c(1 - \cos^2 B - \cos^2 A) = 0.$
- (4) $\frac{\cos^2 B - \cos^2 A}{\cos^2 B \cos^2 A} = \frac{a^4 - b^4}{a^2 b^2}.$
- (5) $a^3 \cos A + b^3 \cos B = abc.$
- (6) $\sin 2A = \frac{2ab}{a^2 + b^2}.$
- (7) $\cos 2A = \frac{b^2 - a^2}{b^2 + a^2}.$
- (8) $\cos \frac{B}{2} = \sqrt{\frac{c+a}{2c}}.$
- (9) $\sin \frac{A}{2} = \sqrt{\frac{c-b}{2c}}.$
- (10) $\tan 2B = \frac{2ab}{a^2 - b^2}.$
- (11) $\sec 2A \tan B = \frac{bc^2}{a(b^2 - a^2)}.$
- (12) $2a^2 \operatorname{cosec} 2B \cot A = c^2.$
- (13) $c^2 \sin(A-B) = a^2 - b^2.$
- (14) $\tan(A-B) = \frac{a^2 - b^2}{2ab}.$
- (15) $\cos(A-B) = \frac{2ab}{c^2}.$
- (16) $\cot \frac{A}{2} = \frac{b+c}{a}.$
- (17) $\tan \frac{A}{2} = \frac{a-b+c}{a+b+c}.$

(ii) Any triangle.

122. The relations connecting angles and sides, which we shall establish as true in the case of **any triangle whatever**, are the following:—

$$(i) \quad \left. \begin{aligned} a &= b \cos C + c \cos B, \\ b &= c \cos A + a \cos C, \\ c &= a \cos B + b \cos A. \end{aligned} \right\}$$

$$(ii) \quad \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

$$(iii) \quad \left. \begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos A, \\ b^2 &= c^2 + a^2 - 2ca \cos B, \\ c^2 &= a^2 + b^2 - 2ab \cos C. \end{aligned} \right\}$$

$$(iv) \quad \left. \begin{aligned} \cos A &= \frac{b^2 + c^2 - a^2}{2bc}, \\ \cos B &= \frac{c^2 + a^2 - b^2}{2ca}, \\ \cos C &= \frac{a^2 + b^2 - c^2}{2ab}. \end{aligned} \right\}$$

$$(v) \quad \left. \begin{aligned} \sin \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}}, \\ \sin \frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{ca}}, \\ \sin \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{ab}}. \end{aligned} \right\}$$

$$(vi) \quad \left. \begin{aligned} \cos \frac{A}{2} &= \sqrt{\frac{s(s-a)}{bc}}, \\ \cos \frac{B}{2} &= \sqrt{\frac{s(s-b)}{ca}}, \\ \cos \frac{C}{2} &= \sqrt{\frac{s(s-c)}{ab}}. \end{aligned} \right\}$$

$$(vii) \quad \left. \begin{aligned} \tan \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}, \\ \tan \frac{B}{2} &= \sqrt{\frac{(s-c)(s-a)}{s(s-b)}}, \\ \tan \frac{C}{2} &= \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}. \end{aligned} \right\}$$

$$(viii) \quad \left. \begin{aligned} \sin A &= \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}, \\ \sin B &= \frac{2}{ca} \sqrt{s(s-a)(s-b)(s-c)}, \\ \sin C &= \frac{2}{ab} \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned} \right\}$$

$$(ix) \quad \left. \begin{aligned} \tan \frac{A-B}{2} &= \frac{a-b}{a+b} \cot \frac{C}{2}, \\ \tan \frac{B-C}{2} &= \frac{b-c}{b+c} \cot \frac{A}{2}, \\ \tan \frac{C-A}{2} &= \frac{c-a}{c+a} \cot \frac{B}{2}. \end{aligned} \right\}$$

123. PROPOSITION I. *To prove that $a = b \cos C + c \cos B$.*

Let ABC be any triangle, and let AD be drawn from A perpendicular to the base BC .

(i) *When D falls between B and C .*

Then $a = BC$

$$= BD + DC$$

$$= BA \cos B + CA \cos C$$

$$= c \cos B + b \cos C.$$

(ii) *When D does not fall between B and C .*

Let D fall to the left of B ; then

$$a = BC$$

$$= DC - DB$$

$$= AC \cos C - AB \cos DBA$$

$$= b \cos C - c \cos (180^\circ - B)$$

$$= b \cos C + c \cos B.$$

(iii) *When one angle is a right angle.*

Let B be a right angle; then

$$a = BC$$

$$= CA \cos C$$

$$= b \cos C$$

$$= b \cos C + c \cos 90^\circ$$

$$= b \cos C + c \cos B.$$

Hence the formula, $a = b \cos C + c \cos B$, is true in all cases.

EXAMPLES 48.

Determine the side a when

(1) $b = 2$, $c = \sqrt{2}$, $B = 45^\circ$, $C = 30^\circ$.

(2) $b = 10$, $c = 10\sqrt{3}$, $B = 30^\circ$, $C = 60^\circ$.

(3) $b = 3\sqrt{2}$, $c = 3\sqrt{3} - 3$, $B = 30^\circ$, $C = 15^\circ$.

(4) $b = 4$, $c = 2\sqrt{3} + 2$, $B = 45^\circ$, $C = 75^\circ$.

(5) $b = 4\sqrt{6}$, $c = 4\sqrt{3} - 4$, $B = 60^\circ$, $C = 15^\circ$.

(6) $b = 6 - 3\sqrt{3}$, $c = 3$, $B = 15^\circ$, $C = 75^\circ$.

124. PROPOSITION II. *To prove that*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Let ABC be any triangle, and let AD be drawn from A perpendicular to the base BC .

(i) *When D falls between B and C .*

Then $AD = AC \sin C$,
 and $AD = AB \sin B$;
 $\therefore AC \sin C = AB \sin B$,
 $\therefore b \sin C = c \sin B$,
 $\therefore \frac{b}{\sin B} = \frac{c}{\sin C}$.

(ii) *When D does not fall between B and C .*

Let D fall to the left of B . Then
 $AD = AC \sin C$,
 and $AD = AB \sin (180^\circ - B) = AB \sin B$;
 $\therefore AC \sin C = AB \sin B$,
 $\therefore b \sin C = c \sin B$,
 $\therefore \frac{b}{\sin B} = \frac{c}{\sin C}$.

(iii) *When one angle is a right angle.*

Let B be a right angle. Then

$AB = AC \sin C$,
 $\therefore AB \sin 90^\circ = AC \sin C$,
 $\therefore AB \sin B = AC \sin C$,
 $\therefore c \sin B = b \sin C$,
 $\therefore \frac{b}{\sin B} = \frac{c}{\sin C}$.

Hence, in all cases, $\frac{b}{\sin B} = \frac{c}{\sin C}$. Similarly $\frac{c}{\sin C} = \frac{a}{\sin A}$.

Hence, in all cases, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (suppose).

NOTE. It will subsequently be proved (Art. 152) that the symbol $2R$ represents the diameter of the circle which circumscribes the triangle.

EXAMPLES 49.

Determine the side a

(1) When $b = 5$, $A = 45^\circ$, $B = 60^\circ$.

(2) When $c = 22$, $A = 45^\circ$, $C = 30^\circ$.

(3) When $b = 19$, $A = 30^\circ$, $B = 75^\circ$.

Determine the side b

(4) When $a = 25$, $B = 30^\circ$, $A = 135^\circ$.

(5) When $c = 39$, $B = 15^\circ$, $C = 120^\circ$.

(6) When $a = 4\frac{1}{2}$, $B = 30^\circ$, $A = 105^\circ$.

125. PROPOSITION III. *To prove that*

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

Let ABC be any triangle, and let CF be drawn from C perpendicular to the base AB .

(i) *When F falls between A and B .*

From Euclid II. 13, we know that

$$\begin{aligned} BC^2 &= AC^2 + AB^2 - 2 \times AB \times AF \\ &= AC^2 + AB^2 - 2 \times AB \times AC \cos A, \end{aligned}$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A.$$

(ii) *When F does not fall between A and B .*

Let F fall to the left of A . Then from Euclid II. 12, we know that

$$\begin{aligned} BC^2 &= AC^2 + AB^2 + 2 \times AB \times AF \\ &= AC^2 + AB^2 + 2 \times AB \times AC \cos(180 - A) \\ &= AC^2 + AB^2 - 2 \times AB \times AC \cos A, \end{aligned}$$

$$\therefore a^2 = b^2 + c^2 - 2bc \cos A.$$

(iii) *When one angle is a right angle.*

Let B be a right angle. Then from Euclid I. 47, we know that

$$\begin{aligned} BC^2 &= AC^2 - AB^2 \\ &= AC^2 + AB^2 - 2 \times AB^2 \\ &= AC^2 + AB^2 - 2 \times AB \times AB \\ &= AC^2 + AB^2 - 2 \times AB \times AC \cos A, \\ \therefore a^2 &= b^2 + c^2 - 2bc \cos A. \end{aligned}$$

Hence, in all cases, $a^2 = b^2 + c^2 - 2bc \cos A$.

126. The formulæ of Proposition III. enable us to determine any side of a triangle when the other sides and the angle between them are given.

Example. Given $a=5$, $b=6$, $C=15^\circ$; find c .

$$\begin{aligned} \text{We have} \quad c^2 &= a^2 + b^2 - 2ab \cos C \\ &= 5^2 + 6^2 - 2 \times 5 \times 6 \cos 15^\circ \\ &= 25 + 36 - 60 \times \frac{\sqrt{3}+1}{2\sqrt{2}} \\ &= 61 - 15\sqrt{2}(\sqrt{3}+1) \\ &= 3.044...; \\ \therefore c &= 1.74, \dots \end{aligned}$$

EXAMPLES 50.

Determine, to two decimal places, the third side in the following cases:—

- (1) $a = \sqrt{3}$, $b = 2$, $C = 30^\circ$.
- (2) $a = \sqrt{2}$, $b = 1$, $C = 45^\circ$.
- (3) $c = 4$, $b = 2$, $A = 60^\circ$.
- (4) $a = \sqrt{2}$, $b = 3$, $C = 135^\circ$.
- (5) $a = 5$, $c = 6$, $B = 120^\circ$.
- (6) $c = 4$, $a = 2\sqrt{3}$, $B = 150^\circ$.
- (7) $c = 2\sqrt{2}$, $b = 1$, $A = 15^\circ$.

127. PROPOSITION IV. *To prove that*

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

This and the formulæ for $\cos B$ and $\cos C$ are those of Proposition III. in another form.

EXAMPLES 51.

Determine $\cos A$ in the following cases:—

(1) $a = \sqrt{3}, b = 2, c = 1.$	(2) $a = 1, b = 1, c = \sqrt{2}.$
(3) $a = 2, b = 1, c = \sqrt{3}.$	(4) $a = 1, b = 2, c = \sqrt{3}.$
(5) $a = 4, b = 2, c = 5.$	(6) $a = 3, b = 6, c = 4.$
(7) $a = 3\frac{1}{2}, b = 2\frac{1}{2}, c = 3.$	(8) $a = 2\frac{1}{4}, b = 1\frac{3}{4}, c = 2.$
(9) $a = 1\frac{1}{3}, b = \frac{2}{3}, c = 1.$	(10) $a = 1\frac{3}{4}, b = 6, c = 5.$

128. PROPOSITION V. *To prove that*

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

From Proposition IV. we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\therefore 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc} = \frac{a^2 - (b-c)^2}{2bc},$$

$$\therefore 2 \sin^2 \frac{A}{2} = \frac{(a-b+c)(a+b-c)}{2bc} = \frac{(2s-2b)(2s-2c)}{2bc},$$

$$\therefore \sin \frac{A}{2} = \pm \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

But, since A is an angle of a triangle, we know that $A < 180^\circ$ and therefore that $\frac{A}{2} < 90^\circ$; hence $\sin \frac{A}{2}$ is positive;

$$\therefore \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

129. PROPOSITION VI. *To prove that*

$$\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

From Proposition IV. we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$\therefore 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc} = \frac{(b+c)^2 - a^2}{2bc},$$

$$\therefore 2 \cos^2 \frac{A}{2} = \frac{(b+c+a)(b+c-a)}{2bc} = \frac{2s(2s-2a)}{2bc},$$

$$\therefore \cos \frac{A}{2} = \pm \sqrt{\frac{s(s-a)}{bc}}.$$

But, since A is an angle of a triangle, we know that $A < 180^\circ$, and therefore that $\frac{A}{2} < 90^\circ$; hence $\cos \frac{A}{2}$ is positive;

$$\therefore \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}.$$

130. PROPOSITION VII. *To prove that*

$$\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

We may proceed thus:—

$$\begin{aligned} \tan^2 \frac{A}{2} &= \frac{1 - \cos A}{1 + \cos A} = \frac{1 - \frac{b^2 + c^2 - a^2}{2bc}}{1 + \frac{b^2 + c^2 - a^2}{2bc}} = \frac{2bc - b^2 - c^2 + a^2}{2bc + b^2 + c^2 - a^2} \\ &= \frac{a^2 - (b-c)^2}{(b+c)^2 - a^2} = \frac{(a+b-c)(a+c-b)}{(b+c+a)(b+c-a)} \\ &= \frac{(2s-2c)(2s-2b)}{2s(2s-2a)} = \frac{(s-c)(s-b)}{s(s-a)}, \end{aligned}$$

$$\therefore \tan \frac{A}{2} = \pm \sqrt{\frac{(s-c)(s-b)}{s(s-a)}}.$$

But, since A is an angle of a triangle, we know that $A < 180^\circ$, and therefore that $\frac{A}{2} < 90^\circ$; hence $\tan \frac{A}{2}$ is positive;

$$\therefore \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

This expression might otherwise be deduced from those for $\sin \frac{A}{2}$ and $\cos \frac{A}{2}$.

131. PROPOSITION VIII. *To prove that*

$$\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

This formula for $\sin A$ may be deduced from the formula for $\cos A$ (Art. 127); otherwise, from Propositions V. and VI. we have

$$\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \text{ and } \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}};$$

$$\begin{aligned} \therefore \sin \frac{A}{2} \cos \frac{A}{2} &= \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}} \\ &= \frac{1}{bc} \sqrt{s(s-a)(s-b)(s-c)}, \end{aligned}$$

$$\therefore \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

NOTE. If we denote the expression $\sqrt{s(s-a)(s-b)(s-c)}$ by Δ , we get

$$\sin A = \frac{2\Delta}{bc}, \quad \sin B = \frac{2\Delta}{ac}, \quad \sin C = \frac{2\Delta}{ab}.$$

The formulæ of Proposition II. may be deduced at once from these.

EXAMPLES 52.

Determine $\sin \frac{A}{2}$, $\cos \frac{A}{2}$, $\tan \frac{A}{2}$ and $\sin A$, when

(1) $a = 2,$	$b = 3,$	$c = 3.$
(2) $a = 8,$	$b = 2,$	$c = 8.$
(3) $a = 3\sqrt{2},$	$b = 3,$	$c = 3.$
(4) $a = 4,$	$b = 2\sqrt{3},$	$c = 2.$
(5) $a = 5\sqrt{3},$	$b = 5,$	$c = 5.$
(6) $a = \sqrt{6},$	$b = \sqrt{3} + 1,$	$c = 2.$

132. PROPOSITION IX. *To prove that*

$$\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

From Art. 124 we have

$$a = 2R \sin A, \text{ and } b = 2R \sin B;$$

$$\therefore \frac{a-b}{a+b} = \frac{2R \sin A - 2R \sin B}{2R \sin A + 2R \sin B} = \frac{\sin A - \sin B}{\sin A + \sin B}$$

$$= \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}} = \cot \frac{A+B}{2} \tan \frac{A-B}{2}$$

$$= \tan \frac{C}{2} \tan \frac{A-B}{2};$$

$$\therefore \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}.$$

133. By means of the formulæ

$$\left. \begin{aligned} \tan \frac{A-B}{2} &= \frac{a-b}{a+b} \cot \frac{C}{2}, \\ A+B &= 180^\circ - C, \end{aligned} \right\}$$

and

we can determine the angles A and B when a , b and C are given. Examples of this kind will be found in Art. 141.

134. The following miscellaneous examples will serve to illustrate the formulæ of this section.

Example i. Prove that

$$(b+c) \cos A + (a+c) \cos B + (a+b) \cos C = a+b+c.$$

The formulæ of Proposition I. are the most suitable to use here.

Thus

$$\begin{aligned} & (b+c) \cos A + (a+c) \cos B + (a+b) \cos C \\ &= b \cos A + c \cos A + a \cos B + c \cos B + a \cos C + b \cos C \\ &= (b \cos C + c \cos B) + (c \cos A + a \cos C) + (a \cos B + b \cos A) \\ &= a+b+c. \end{aligned}$$

Example ii. Prove that $b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} = s$.

(a) We may use the formulæ of Art. 60, thus:—

$$\begin{aligned} b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} &= \frac{b}{2} (1 + \cos C) + \frac{c}{2} (1 + \cos B) \\ &= \frac{1}{2} (b + b \cos C + c + c \cos B) \\ &= \frac{1}{2} (b + c + b \cos C + c \cos B) \\ &= \frac{1}{2} (b + c + a) \\ &= s. \end{aligned}$$

(β) We may use the formulæ of Art. 129, thus:—

$$\begin{aligned} b \cos^2 \frac{C}{2} + c \cos^2 \frac{B}{2} &= b \times \frac{s(s-c)}{ab} + c \times \frac{s(s-b)}{ac} \\ &= \frac{s(s-c)}{a} + \frac{s(s-b)}{a} \\ &= \frac{s}{a} (2s - c - b) = \frac{s}{a} \times a \\ &= s. \end{aligned}$$

Example iii. Prove that $\frac{a-b}{c} \cos \frac{C}{2} = \sin \frac{A-B}{2}$.

$$\begin{aligned} \frac{a-b}{c} \cos \frac{C}{2} &= \frac{2R \sin A - 2R \sin B}{2R \sin C} \times \cos \frac{C}{2} \\ &= \frac{\sin A - \sin B}{\sin C} \times \cos \frac{C}{2} \\ &= \frac{2 \cos \frac{A+B}{2} \sin \frac{A-B}{2}}{2 \sin \frac{C}{2} \cos \frac{C}{2}} \times \cos \frac{C}{2} \\ &= \sin \frac{A-B}{2}. \end{aligned}$$

EXAMPLES 53.

Prove that

$$(1) \quad \frac{\sin A}{\sin(A+B)} = \frac{a}{c}. \quad (2) \quad \frac{\sin A}{a} = \frac{\sin B + \sin C}{b + c}.$$

$$(3) \quad b(a \cos C - c \cos A) = a^2 - c^2.$$

$$(4) \quad a \cos B - b \cos A = \frac{a^2 - b^2}{c}.$$

$$(5) \quad 2bc \cos A + 2ac \cos B + 2ab \cos C = a^2 + b^2 + c^2.$$

$$(6) \quad a(\sin B - \sin C) + b(\sin C - \sin A) + c(\sin A - \sin B) = 0.$$

$$(7) \quad \frac{\sin(A-C)}{\sin B} = \frac{a^2 - c^2}{b^2}. \quad (8) \quad \frac{c - b \cos A}{b - c \cos A} = \frac{\cos B}{\cos C}.$$

- (9) $\frac{b}{c} (\cot A + \cot B) = \operatorname{cosec} A.$
- (10) $a \cos^2 \frac{C}{2} + c \cos^2 \frac{A}{2} = s.$
- (11) $4 \left(bc \cos^2 \frac{A}{2} + ca \cos^2 \frac{B}{2} + ab \cos^2 \frac{C}{2} \right) = (a + b + c)^2.$
- (12) $\tan \frac{B}{2} \cot \frac{C}{2} = \frac{a + b - c}{a - b + c}.$
- (13) $b \cos A \cot B + c \cot C = a \sin B.$
- (14) $a (b^2 + c^2) \cos A + b (c^2 + a^2) \cos B + c (a^2 + b^2) \cos C = 3abc.$
- (15) $a \sin (B - C) + b \sin (C - A) + c \sin (A - B) = 0.$
- (16) $a \cos A \cos 2B + b \cos B \cos 2A + c \cos C = 0.$
- (17) $\frac{a + b}{c} \sin \frac{C}{2} = \cos \frac{A - B}{2}.$
- (18) $(a - b) \cos \frac{A - B}{2} \cos \frac{C}{2} = (a + b) \sin \frac{A - B}{2} \sin \frac{C}{2}.$
- (19) $(a + b) \times 2 \sin^2 \frac{C}{2} = c (\cos A + \cos B).$
- (20) $(a - b) (1 + \cos C) = c (\cos B - \cos A).$
- (21) $(a^2 - b^2) \cos C = c (b \cos B - a \cos A).$
- (22) $(b^2 - c^2) \sin A = a^2 \sin (B - C).$
- (23) $\frac{a^2 - b^2}{ab} \operatorname{cosec} C = \cot B - \cot A.$
- (24) $(b^2 - c^2) \cot A + (c^2 - a^2) \cot B + (a^2 - b^2) \cot C = 0.$
- (25) $\frac{b - c}{a} \cos^2 \frac{A}{2} + \frac{c - a}{b} \cos^2 \frac{B}{2} + \frac{a - b}{c} \cos^2 \frac{C}{2} = 0.$

EXAMPLES 54.

- (1) If $\sin^2 C = \sin^2 A + \sin^2 B$, prove that the triangle is right-angled.
- (2) If $a \cos A = b \cos B$, prove that the triangle is either right-angled or isosceles.
- (3) If $a \cos B = b \cos A$, what relation between the angles can be inferred?

(4) If $(a^2 + b^2) \cos 2A = b^2 - a^2$, prove that the triangle must be right-angled.

(5) If $\frac{\cos A}{a} = \frac{\cos B}{b} = \frac{\cos C}{c}$, prove that the triangle is equilateral.

(6) Prove that $8 \cos A \cos C = 3 \cos B$, when $a = \sqrt{5}$, $b = 2$, $c = \sqrt{3}$.

(7) If a triangle be formed having for its sides

$$c, (a+b) \sin \frac{C}{2}, \text{ and } (a-b) \cos \frac{C}{2},$$

shew that it is right-angled.

(8) If $A = 45^\circ$ and $B = 60^\circ$, prove that $2c = a(1 + \sqrt{3})$.

(9) If $A = 40^\circ$ and $C = 80^\circ$, prove that $a + c = 2b \cos 20^\circ$.

(10) If $\sin A : \sin B : \sin C = 13 : 14 : 15$, find $\cos A$.

(11) Prove that the perpendicular from A on BC

$$= \frac{b^2 \sin C + c^2 \sin B}{b + c}.$$

(12) If D be the middle point of BC, and X the foot of the perpendicular from A on BC, prove that

$$DX = \frac{b^2 - c^2}{2a}.$$

(13) If AD be the median through the point A, i.e., the line which joins A to the middle point of BC, prove that

$$AD^2 = \frac{1}{2} \left(b^2 + c^2 - \frac{a^2}{2} \right).$$

(14) If AD, BE, CF be the three medians, prove that

$$AD^2 + BE^2 + CF^2 = \frac{3}{4} (a^2 + b^2 + c^2).$$

(15) If $\cot A$, $\cot B$, $\cot C$ be in Arithmetical Progression, prove that a^2 , b^2 , c^2 are also in Arithmetical Progression.

(16) If A, B, C be the angles of a triangle, prove that the value of $\sin B \sin C \cos A + \sin^2 A$ will remain unchanged when any two of the angles A, B, C are interchanged.

(17) Shew that $\frac{\sin 2A + \sin 2B + \sin 2C}{\sin A + \sin B + \sin C} = 8 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$.

(18) Shew that $\frac{\sin B + \sin C - \sin A}{\sin B + \sin C + \sin A} = \tan \frac{B}{2} \tan \frac{C}{2}$.

XIII. THE SOLUTION OF TRIANGLES.

135. When a sufficient number of the six elements of a triangle are given, and we use the various properties which have been established to determine the remaining elements, we are said to **solve the triangle**. The calculations involved will be simplified in some cases by the use of logarithms, by which we are enabled to substitute addition and subtraction for multiplication and division. Any three of the six elements of a triangle will be enough for the purpose, provided that **at least one** of the three is a **side**. Three *angles* will not suffice, for when two angles are given, the third is necessarily known; of three such data only two therefore can be considered independent.

All plane triangles whatever may be classed under one or other of two heads;—(i) Right-angled triangles; (ii) Triangles which are not right-angled. We will consider these classes in turn.

The Solution of right-angled triangles.

136. Right-angled triangles of which one angle is 30° , 45° , or any one of those whose ratios have been specially determined, were briefly considered in Art. 41. We have still however to consider cases involving other angles.

(a) When an acute angle and a side are given.

137. (i) Let A and a be the given elements.
Then

$$B = 90^\circ - A,$$

$$b = a \cot A,$$

and
$$c = a \operatorname{cosec} A.$$

If in these equations we substitute the known values of A , a , $\cot A$, and $\operatorname{cosec} A$, we obtain the required values of B , b and c .

(ii) If A and b be the given elements, then

$$\left. \begin{aligned} B &= 90^\circ - A, \\ a &= b \tan A, \\ c &= b \sec A. \end{aligned} \right\}$$

(iii) If A and c be the given elements, then

$$\left. \begin{aligned} B &= 90^\circ - A, \\ a &= c \sin A, \\ b &= c \cos A. \end{aligned} \right\}$$

We may from these equations obtain the required elements, either by direct substitution for A and the given side, or by using logarithms. By direct substitution the process is sufficiently simple.

EXAMPLES 55.

Solve a right-angled triangle in the following cases, the right angle being at C :—

(1) $c = 40$, $A = 75^\circ$.

Given $\sin 75^\circ = \cdot 9659258$,

$\cos 75^\circ = \cdot 2588190$.

(2) $c = 22$, $A = 15^\circ 47'$.

Given $\sin 15^\circ 47' = \cdot 2720003$,

$\cos 15^\circ 47' = \cdot 9622972$.

(3) $b = 14$, $A = 47^\circ 5'$.

Given $\tan 47^\circ 5' = 1\cdot 0755006$,

$\sec 47^\circ 5' = 1\cdot 4685713$.

(4) $b = 3\cdot 25$, $A = 60^\circ$.

Given $\tan 60^\circ = 1\cdot 7320508$,

$\sec 60^\circ = 2$.

(5) $a = 25$, $A = 36^\circ 10'$.

Given $\cot 36^\circ 10' = 1\cdot 3679959$,

$\operatorname{cosec} 36^\circ 10' = 1\cdot 6945244$.

(6) $a = 15$, $A = 78^\circ 34'$.

Given $\cot 78^\circ 34' = \cdot 2022409$,

$\operatorname{cosec} 78^\circ 34' = 1\cdot 0202457$.

(β) When two of the three sides are given.

138. (i) If a and b be the given elements, we have the following equations :—

$$\tan A = \frac{a}{b}, \text{ whence we obtain } A;$$

$$B = 90^\circ - A, \text{ whence we can then obtain } B;$$

and $c = a \operatorname{cosec} A$, whence we get c .

Using logarithms, the first and third equations become

$$L \tan A = 10 + \log a - \log b,$$

and $\log c = \log a - L \sin A + 10.$

(ii) If a and c be the given elements, we have

$$\sin A = \frac{a}{c}, \text{ whence we obtain } A;$$

$$B = 90^\circ - A, \text{ whence we then obtain } B;$$

and $b = c \cos A,$

or

$$b = \sqrt{(c+a)(c-a)},$$

} from either of which
we may obtain b .

Using logarithms, the first and last equations become

$$L \sin A = 10 + \log a - \log c,$$

and $\log b = \log c + L \cos A - 10,$

or

$$\log b = \frac{1}{2} \{ \log (c+a) + \log (c-a) \}.$$

EXAMPLES 56.

Determine the other elements in the following cases, C being the right angle:—

(1) $a = 3, b = 4.$

Given $\tan 36^\circ 52' = \cdot 7499119$; diff. for $1' = \cdot 0004546.$

(2) $c = 23, b = 10.$

Given $\log 23 = 1\cdot 3617278,$

$\sec 64^\circ 13' = 2\cdot 2990134,$ diff. for $1' = \cdot 0013854.$

(3) $a = 5, b = 7\cdot 5.$

Given $\tan 33^\circ 41' = \cdot 6664969,$ diff. for $1' = \cdot 0004202,$

$\operatorname{cosec} 33^\circ 41' = 1\cdot 8030935,$ diff. for $1' = \cdot 0007865.$

(4) $a = 37, b = 9\cdot 25.$

Given $\tan 75^\circ 57' = 3\cdot 9959223,$ diff. for $1' = \cdot 0049413,$

$\sec 75^\circ 57' = 4\cdot 1191498,$ diff. for $1' = \cdot 0047937.$

The solution of triangles which are not right-angled.

(a) When three sides are given.

139. We have in this case the following formulæ at our disposal for the determination of the angle A :—

$$(i) \quad \cos A = \frac{b^2 + c^2 - a^2}{2bc}; \quad (ii) \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}};$$

$$(iii) \quad \cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}; \quad (iv) \quad \sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}};$$

$$(v) \quad \sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}.$$

Which of these formulæ is to be used in any question will generally depend to some extent upon the data given in the question: but if the choice is not restricted, that formula should be selected which promises to lead to the required result with least calculation.

If the values of $a, b,$ and c are **simple**, and such as can be multiplied, &c., mentally, it may seem best to determine $\cos A$ by

means of (i), and thence to determine A directly; but when these values are large or complex, the multiplications and divisions involved in (i) become troublesome, and moreover cannot be avoided by using logarithms.

If the values of a , b , and c are not suitable for (i) the best formula to use is (ii). Logarithms can be used with it, and moreover the same logarithms will be required for one angle as for another, namely, those of s , $s-a$, $s-b$, and $s-c$.

$$\text{Thus} \quad \tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\text{and} \quad \tan \frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}};$$

$$\therefore L \tan \frac{A}{2} = 10 + \frac{1}{2} \{ \log(s-b) + \log(s-c) - \log s - \log(s-a) \},$$

$$\text{and} \quad L \tan \frac{B}{2} = 10 + \frac{1}{2} \{ \log(s-a) + \log(s-c) - \log s - \log(s-b) \}.$$

When A and B have been determined in this way, we can obtain C from the relation, $C = 180^\circ - A - B$.

NOTE. If only *one* angle A is required, it will generally be best to reduce the expression for $\tan \frac{A}{2}$ to its lowest terms, before using logarithms; but if *two* angles, A and B , have to be determined, this should not usually be done.

Example. Given $a=2364$, $b=3280$, $c=4120$; find the angles.

We have $2s=2364+3280+4120=9764$,

$$\therefore s=4882, \quad s-b=1602, \quad s-a=2518, \quad s-c=762,$$

$$\therefore \tan \frac{A}{2} = \sqrt{\frac{1602 \times 762}{4882 \times 2518}},$$

$$\begin{aligned} \therefore L \tan \frac{A}{2} &= 10 + \frac{1}{2} (\log 1602 + \log 762 - \log 4882 - \log 2518) \\ &= 10 + \frac{1}{2} \left\{ \begin{array}{l} 3.2046625 - 3.6885978 \\ + 2.8819550 - 3.4010557 \end{array} \right\} \text{ [from Tables]} \\ &= 10 + \frac{1}{2} \left\{ \begin{array}{l} 6.0866175 \\ - 7.0896535 \end{array} \right\} \\ &= 10 - \frac{1}{2} (1.0030360) = 10 - .5015180 \\ &= 9.4984820. \end{aligned}$$

But $L \tan 17^\circ 29' = 9.4982816$, and diff. for $60'' = .0004407$;

$\therefore$ no. of seconds to be added to $17^\circ 29' = \frac{2.004}{4.107} \times 60 = 27.3$;

$$\therefore \frac{A}{2} = 17^\circ 29' 27''.3;$$

$$\therefore A = 34^\circ 58' 54''.6.$$

Similarly we may find B , and thence C .

NOTE. To determine the angles of a triangle it is not necessary that the actual lengths of the three sides should be given. It is sufficient to know the *ratios* which the lengths bear one to another. For example, if $a : b : c = 2 : 5 : 6$, we may put $2k, 5k, 6k$ for a, b, c , respectively, in the formula for $\tan \frac{A}{2}$, and it will be seen that the k 's will disappear by cancelling.

EXAMPLES 57.

(1) If $a = 114$, $b = 101$, $c = 25$, find $\tan \frac{A}{2}$.

(2) The sides of a triangle are proportional to 5, 7 and 9; find the cosine of the greatest angle.

(3) If the sides of a triangle be 3, $3\frac{1}{2}$ and 5, shew from the formula for the cosine of an angle that the triangle is obtuse-angled.

(4) If the sides of a triangle be 5, 7 and 8, shew that one of the angles is 60° .

(5) If the sides of a triangle be 4, 5 and 6, find the greatest angle; given $\cos 82^\circ 49' = .125$.

(6) If the sides a, b, c , respectively, be 2, $\sqrt{6}$, and $1 + \sqrt{3}$, find A .

(7) Find the smallest angle of the triangle whose sides are 25, 30 and 35; given $\sin 45^\circ 35' 5'' = .714286$.

(8) If $a = 18$, $b = 24$, $c = 30$, find $\sin A$.

(9) The sides of a triangle are 10, 15 and 20; find the largest angle. Given $\log 2$; also

$$L \sin 52^\circ 14' = 9.8979082, \text{ diff. for } 1' = .0000978.$$

(10) If $a = 297$, $b = 351$, $c = 432$, find A . Given $\log 2$, $\log 3$ and $\log 7$; also $L \cot 21^\circ 31' = 10.4042321$,
 $L \cot 21^\circ 32' = 10.4038620$ }

(11) If the sides be 14, 16 and 18, find the least angle.

$$\left. \begin{array}{l} \text{Given } L \tan 24^\circ 5' 40'' = 9.6505069 \\ L \tan 24^\circ 5' 50'' = 9.6505634 \end{array} \right\}, \text{ and } \log 2.$$

(12) If $a : b : c = 4 : 5 : 6$, find B.

$$\left. \begin{array}{l} \text{Given } L \cos 27^\circ 53' = 9.9464040 \\ L \cos 27^\circ 54' = 9.9463371 \end{array} \right\}, \text{ and } \log 2.$$

(13) If $a = 16$, $b = 36$, $c = 44$, find A.

$$\begin{array}{l} \text{Given } \log 32 = 1.5051500; \quad \log 33 = 1.5185139; \\ L \cos 10^\circ 1' = 9.9933292, \quad \text{diff. for } 1' = .0000224. \end{array}$$

(β) **When two angles and a side are given.**

140. If the given elements be A, B and c , we may determine C, a and b thus:—

$$(i) \quad C = 180^\circ - A - B.$$

$$(ii) \quad \frac{a}{c} = \frac{\sin A}{\sin C}, \quad \therefore \log a = \log c + L \sin A - L \sin (A + B).$$

$$(iii) \quad \frac{b}{c} = \frac{\sin B}{\sin C}, \quad \therefore \log b = \log c + L \sin B - L \sin (A + B).$$

Example. Given $A = 17^\circ 28' 5''$, $B = 133^\circ 41' 13''$, and $b = 28$; find C, a and c .

$$(i) \quad C = 180^\circ - 17^\circ 28' 5'' - 133^\circ 41' 13'' = 28^\circ 50' 42''.$$

$$(ii) \quad \frac{a}{b} = \frac{\sin A}{\sin B},$$

$$\begin{aligned} \therefore \log a &= \log b + L \sin A - L \sin B \\ &= \log 28 + L \sin 17^\circ 28' 5'' - L \sin 133^\circ 41' 13'' \\ &= \log 28 + L \sin 17^\circ 28' 5'' - L \sin 46^\circ 18' 47''. \end{aligned}$$

We calculate the required logarithms with the help of the Tables, and we then proceed to find a as in former cases. The side c may be obtained in a similar way.

EXAMPLES 58.

(1) If $A = 135^\circ$, $B = 15^\circ$ and $c = 19$, find the other sides.
Given $\sin 15^\circ = .2588190$, and $\sin 135^\circ = .7071068$.

(2) If $B = 45^\circ$, $C = 10^\circ$ and $a = 200$ feet, find b .

$$\begin{array}{l} \text{Given } \log 172.64 = 2.2371414, \log 172.65 = 2.2371666, \\ L \sin 55^\circ = 9.9133645, \text{ and } \log 2. \end{array}$$

(3) The base of a triangle being 7 ft., and the base angles $129^{\circ} 23'$ and $38^{\circ} 36'$, find the length of the longest side. Given

$$L \sin 50^{\circ} 37' = 9.8881335; L \sin 12^{\circ} 11' = 9.3243657;$$

$$\log 2.5636 = 1.4088503, \text{ diff.} = .0000169; \log 7.$$

(4) The angles of a triangle are 40° , 60° , and 80° respectively, and the greatest side is 22 ft.; find the other sides. Given

$$L \sin 40^{\circ} = 9.8080675; L \sin 80^{\circ} = 9.9933515;$$

$$\log 22 = 1.3424227; \log 14359 = 4.1571242, \text{ diff.} = .0000302;$$

$$L \sin 60^{\circ} = 9.9375306; \log 19346 = 4.2865912, \text{ diff.} = .0000224.$$

(7) When two sides and the included angle are given.

141. If a , b , and C be the given elements, we may determine c , A and B , thus:—

$$(i) \text{ We have } \tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2},$$

$$\therefore L \tan \frac{A-B}{2} = \log(a-b) - \log(a+b) + L \cot \frac{C}{2}.$$

Whence we determine the angle $\frac{A-B}{2}$.

$$\text{Also } \frac{A+B}{2} = \frac{180^{\circ} - C}{2} = 90^{\circ} - \frac{C}{2}; \text{ whence we get } \frac{A+B}{2}.$$

From these two results we find A and B .

The student should generally reduce $\frac{a-b}{a+b}$ to its lowest terms, before introducing logarithms.

$$(ii) \text{ Also, } \frac{c}{a} = \frac{\sin C}{\sin A}, \therefore \log c = \log a + L \sin C - L \sin A;$$

whence we may determine c .

Example. If $b=585$, $c=473$, and $A=45^{\circ} 20' 30''$, find (i) the angles B and C , and (ii) the side a .

$$(i) \text{ We have } \tan \frac{B-C}{2} = \frac{b-c}{b+c} \cot \frac{A}{2} = \frac{56}{529} \cot 22^{\circ} 40' 15'',$$

$$\therefore L \tan \frac{B-C}{2} = \log 56 - \log 529 + L \cot 22^{\circ} 40' 15''$$

$$\left[\begin{array}{l} L \cot 22^{\circ} 40' = 10.3792128, \\ \text{diff. for } 15'' = .0000888, \\ \therefore L \cot 22^{\circ} 40' 15'' = 10.3791240. \end{array} \right] \begin{array}{l} = 1.7481880 + 10.3791240 - 2.7234557 \\ = 9.4038563. \end{array} \quad [\text{Tables}]$$

Also $L \tan 14^\circ 13' = 9.4037182$; diff. for $1' = 5304$;

$$\therefore \frac{B-C}{2} = 14^\circ 13' + \frac{1381}{5304} \times 60'' = 14^\circ 13' 15''.62;$$

and

$$\therefore \left. \begin{aligned} B-C &= 28^\circ 26' 31''.24 \\ B+C &= 134^\circ 39' 30'' \end{aligned} \right\};$$

$$\therefore \left. \begin{aligned} B &= 81^\circ 33' 0''.62 \\ C &= 53^\circ 6' 29''.38 \end{aligned} \right\}.$$

(ii) To find the side a we now have

$$\begin{aligned} \log a &= \log b + L \sin A - L \sin B \\ &= \log 585 + L \sin 45^\circ 20' 30'' - L \sin 81^\circ 33'.0''.62 \\ &= \left. \begin{aligned} &2.7671559 \\ &+ 9.8519970 \\ &+ .0000624 \end{aligned} \right\} - \left\{ \begin{aligned} &9.9952597 \\ &+ .0000002 \end{aligned} \right. \quad [\text{Tables}] \\ &= 12.6192153 - 9.9952599 \\ &= 2.6239554; \end{aligned}$$

$$\therefore a = 420.68, \quad \text{from the Tables.}$$

EXAMPLES 59.

- (1) If $b = 11$, $c = 8$, and $A = 60^\circ$, find a to 4 decimal places.
- (2) If $b = 2$, $c = 3$, and $A = \cos^{-1} \frac{1}{3}$, find a , $\cos B$ and $\cos C$.
- (3) If $a = 3$, $b = 5$, and $C = 120^\circ$, find c , $\cos A$ and $\cos B$.
- (4) If $a = 5$, $b = 4$, and $\tan \frac{A+B}{2} = 3\sqrt{3}$, find $A - B$.
- (5) If $\tan \frac{B-C}{2} = \frac{1}{21} \sqrt{7}$, $b = 50$ and $c = 40$, find $\tan \frac{A}{2}$.
- (6) If $a : b = 2 + \sqrt{3} : 1$, and $C = 60^\circ$, find A and B .
- (7) If $a = 4$, $b = 1$, and $C = 60^\circ$, find c , $\sin B$ and $\tan B$.
- (8) If $c = 7$, $a = 3\frac{1}{2}$, and $B = 60^\circ$, find b , A and C .
- (9) If $a = 20$, $b = 10$, and $C = 60^\circ$, find c , A and B .
- (10) If $a = 65$, $b = 63$, and $C = 52^\circ 22'$, find A and B .

Given $L \cot 26^\circ 11' = 10.30830$
and $L \tan 1^\circ 49' 12'' = 8.50212$ } ; also $\log 2$.

- (11) If $a = 70$, $b = 35$, and $C = 36^\circ 52' 12''$, find A and B .

Given $\log 3$, and $L \cot 18^\circ 26' 6'' = 10.4771213$.

(12) If $a : b = 21 : 11$, and $C = 34^\circ 42' 30''$, find A and B .

Given $\log 2$, and $L \cot 17^\circ 21' 15'' = 10.50515$.

(13) If $a = 42\frac{1}{2}$, $b = 37\frac{1}{2}$, and $C = 70^\circ$, find A and B .

Given $L \cot 35^\circ = 10.154773$, $L \tan 5^\circ 6' = 8.950596$,

$\log 2$, and $L \tan 5^\circ 7' = 8.952021$.

(14) If $a = 189$, $b = 61$, and $C = 60^\circ$, find A and B .

Given $L \tan 41^\circ 34' = 9.9478265$
 $L \tan 41^\circ 35' = 9.9480810$ } ; also $\log 2$ and $\log 3$.

(15) If $a = 384$, $b = 330$, and $C = 90^\circ$, find A and B .

Given $L \tan 4^\circ 19' = 8.8778437$
diff. for $1' = .0016799$ } ; also $\log 3$, $\log 7$ and $\log 17$.

(16) If $a = 95$, $b = 72$, and $C = 37^\circ 33' 20''$, find A and B .

Given $\log 23 = 1.3617278$, $L \tan 22^\circ 3' = 9.6074997$,

$\log 167 = 2.2227165$, $L \tan 22^\circ 4' = 9.6078627$,

$L \cot 18^\circ 46' 40'' = 10.4685275$.

(δ) **The ambiguous case;—Given two sides and the angle opposite to one of them.**

142. This case calls for most careful consideration. There may be *no* triangle at all, or *one* triangle, or *two* triangles, according to the conditions which the given values satisfy.

Let a , b and A be the given elements, and let $\hat{C}AX = A$, $AC = b$, and $LM = a$; and let CD be drawn from C perpendicular to AX .

We have then two angular points A and C of the required triangle already, and it remains only to determine the position of the third angular point B . Of this position of B we know two facts:—(i) that it lies on AX , and (ii) that it is at the distance LM from C . Consequently the point B must be where AX meets the circle described with centre C and radius LM . We have however three possible cases to consider:—

(i) If $LM < CD$, the circle *does not meet* AX at all, and therefore **no** triangle with the given elements can be constructed.

(ii) If $LM = CD$, the circle *touches* AX , and therefore **one** triangle, namely ACD , can be constructed.

(iii) If $LM > CD$, the circle will cut the line AX in *two* points B_1 and B_2 . These points will be on the *same* side of A , if $LM < AC$, and on *opposite* sides of A , if $LM > AC$.

Consequently, when $LM > CD$, but $< AC$, as in the left-hand figure, the construction gives us **two** triangles, AB_1C and AB_2C , both of which have the three given elements a , b , and A .

On the other hand, when $LM > CD$, and $> AC$, as in the right-hand figure, the triangle ACB_2 has the given elements a and b , but the angle opposite to the side CB_2 , or a , is $180^\circ - A$ instead of A . In this case therefore only one of the triangles, namely, ACB_1 , has all three of the given elements.

The perpendicular CD is equal to $b \sin A$, hence we may state the result of the above investigation briefly thus:—**When the given elements are two sides a , b , and an angle A opposite to one of them, it will be possible to describe **TWO****

dissimilar triangles with these elements, when a , b and A are such that $a > b \sin A$, and $< b$, i.e., when a lies in magnitude between b and $b \sin A$.

In consequence of the data leading to two solutions, this is called the **ambiguous case**.

143. To solve a triangle in the ambiguous case.

Let a , b and A be the given elements, such that

$$a > b \sin A, \text{ and } < b.$$

$$\begin{aligned} \text{(i) We have } \sin B &= \frac{b}{a} \sin A, \\ \text{or } L \sin B &= \log b - \log a + L \sin A. \end{aligned} \quad \left. \vphantom{\begin{aligned} \text{(i) We have } \sin B &= \frac{b}{a} \sin A, \\ \text{or } L \sin B &= \log b - \log a + L \sin A. \end{aligned}} \right\}$$

From these relations we obtain the value of $\sin B$, or of $L \sin B$; and thence we obtain *two* possible values of B , namely,

$$B_1 (\text{less than } 90^\circ), \text{ and } 180^\circ - B_1.$$

(ii) $C = 180^\circ - A - B$; hence we obtain *two* values of C , namely,

$$\begin{aligned} &180^\circ - A - B_1, \\ \text{and } &180^\circ - A - (180^\circ - B_1), \end{aligned} \quad \left. \vphantom{\begin{aligned} &180^\circ - A - B_1, \\ \text{and } &180^\circ - A - (180^\circ - B_1), \end{aligned}} \right\} \text{ i.e., } \left\{ \begin{aligned} &180^\circ - A - B_1, \\ \text{and } &B_1 - A. \end{aligned} \right.$$

(iii) The third side c may be obtained from the formula

$$\frac{c}{a} = \frac{\sin C}{\sin A}, \text{ whence } \log c = \log a + L \sin C - L \sin A.$$

Since $\sin C$ has two different values, this formula leads to *two* different values of c .

NOTE. When the given elements are such that only one of these solutions is possible, i.e., when $a > b$, we determine $\sin B$ as above, and we select only that value of B which is less than 90° . In this case A may be either acute or *obtuse*.

144. The two values of the third side c in terms of a , b , and A , may also be obtained algebraically direct from the equation

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$\text{Thus } c^2 - 2bc \cos A + b^2 - a^2 = 0;$$

$$\therefore c = \frac{2b \cos A \pm \sqrt{4b^2 \cos^2 A - 4b^2 + 4a^2}}{2}$$

$$= b \cos A \pm \sqrt{a^2 - b^2 \sin^2 A}.$$

Hence, if c_1 and c_2 be the two values of the third side, we have

$$\left. \begin{aligned} c_1 + c_2 &= 2b \cos A, \\ c_1 c_2 &= b^2 - a^2. \end{aligned} \right\}$$

and

NOTE. From the expression under the radical in the value of c , we may also determine the conditions for the existence of two triangles, one only, or none.

EXAMPLES 60.

State the number of solutions in the following cases:—

- (1) $A = 30^\circ$, $b = 8$, $a = 4$. (2) $A = 30^\circ$, $b = 8$, $a = 6$.
 (3) $A = 30^\circ$, $b = 8$, $a = 3$. (4) $A = 30^\circ$, $b = 8$, $a = 9$.

In the following four cases, solve the triangle without using logarithms:—

- (5) $A = 30^\circ$, $c = 17\frac{1}{2}$, $a = 8\frac{1}{4}$.
 (6) $A = 120^\circ$, $a = \sqrt{3}$, $b = 1$.
 (7) $A = 105^\circ$, $a = \sqrt{3} + 1$, $c = \sqrt{6}$.
 (8) $A = 30^\circ$, $a = 1$, $c = \sqrt{3}$.
 (9) If $a = 35$, $c = 20$, $A = 107^\circ 25'$, find C . Given $\log 2$;
 $\log 3 \cdot 5 = \cdot 5440680$; $L \sin 33^\circ 2' = 9 \cdot 7364976$,
 $L \sin A = 9 \cdot 9796182$; $L \sin 33^\circ 3' = 9 \cdot 7366918$.
 (10) If $a = 42$, $b = 80$ and $B = 60^\circ$, find A .
 Given $L \sin 27^\circ 2' 30'' = 9 \cdot 6576661$, } $\log 2$, $\log 3$ and $\log 7$.
 $L \sin 27^\circ 2' 40'' = 9 \cdot 6577073$, }
 (11) If $a = 8$, $b = 7$, $A = 120^\circ$, find B and C .
 Given $L \sin 60^\circ = 9 \cdot 9375306$, $L \sin 49^\circ 16' = 9 \cdot 8795287$,
 $\log 7$ and $\log 2$, $L \sin 49^\circ 17' = 9 \cdot 8796375$.

XIV. THE AREA OF A TRIANGLE.

145. We learn from Euclid that the area of a *triangle* is half that of a *rectangle* on the same base and of the same height.

But the area of a *rectangle* is obtained by multiplying the number of units of length in the *base* by the number of units of length in the *height*; the result being the number of units of area in the *area*.

Consequently, if a be the number of units of length in the base BC of the rectangle $BCEF$, and h be the number of units of length in the height AD , the area of the rectangle will be measured by ah , and the area of the triangle BAC by $\frac{1}{2}ah$;

$$\therefore \text{area of } \triangle ABC = \frac{1}{2}ah = \frac{1}{2}ab \sin C.$$

Similarly the area may be represented by

$$\frac{1}{2}bc \sin A, \text{ and also by } \frac{1}{2}ac \sin B;$$

and, generally, the **area of a triangle** is measured by **half the product of any two sides and the sine of the included angle**. To denote this area we will use the symbol Δ .

146. From the formula $\Delta = \frac{1}{2}bc \sin A$ we may deduce an expression for Δ in terms of s and the three sides.

$$\text{Thus} \quad \Delta = \frac{1}{2}bc \sin A$$

$$\begin{aligned} &= \frac{1}{2}bc \times \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)} \quad (\text{Art. 131}) \\ &= \sqrt{s(s-a)(s-b)(s-c)}. \end{aligned}$$

147. It is necessary to **remember** the expressions which have been obtained for the area of a triangle, namely,

$$\Delta = \frac{1}{2}bc \sin A = \frac{1}{2}ac \sin B = \frac{1}{2}ab \sin C,$$

and

$$\Delta = \sqrt{s(s-a)(s-b)(s-c)}.$$

Other expressions may be deduced from these, but are of less importance. For example, to suit the case in which two angles A , B , and a side c , are given, the formula $\Delta = \frac{1}{2}bc \sin A$ may be transformed thus:—

$$\Delta = \frac{1}{2}bc \sin A = \frac{1}{2} \frac{c \sin B}{\sin C} \times c \sin A = \frac{c^2 \sin A \sin B}{2 \sin(A+B)}.$$

It may be noticed too that all these formulæ for the area of a triangle are suitable for use with logarithms: thus—

$$\log \Delta = \log b + \log c - \log 2 + L \sin A - 10,$$

and $\log \Delta = \frac{1}{2} \{ \log s + \log (s-a) + \log (s-b) + \log (s-c) \}.$

EXAMPLES 61.

Find the area of the triangle in the following cases. Given

- (1) $a=8, b=3, C=90^\circ.$ (2) $a=6, b=7, C=90^\circ.$
 (3) $a=5, b=7, C=30^\circ.$ (4) $a=12, b=6, C=150^\circ.$
 (5) $a=14, c=10, B=120^\circ.$ (6) $b=3, c=21, A=45^\circ.$
 (7) $a=12, b=15, c=9.$ (8) $a=13, b=14, c=15.$
 (9) $A=30^\circ, B=60^\circ, c=12.$ (10) $A=60^\circ, B=45^\circ, c=16.$
 (11) If the area of an equilateral triangle be 596 square yards, find its sides.
 (12) From the formulæ in Art. 147 deduce the following expressions for the area of a triangle:—

(i) $\frac{c^2}{2(\cot A + \cot B)}.$

(ii) $\frac{1}{4}(c^2 \sin 2B + b^2 \sin 2C).$

(iii) $\frac{2s^2 \sin A \sin B \sin C}{(\sin A + \sin B + \sin C)^2}.$

(iv) $\frac{2abc}{a+b+c} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$

XV. PROBLEMS INVOLVING THE SOLUTION OF TRIANGLES.

148. We will now consider some practical questions which depend for their solution upon the methods which we have already given for solving triangles under various conditions, but we may remark again here that no very definite rules can be laid down for the solution of these problems, beyond two;—That the student should draw the diagrams very carefully and neatly, marking all the data, and that he should have readily at his command the various formulæ which have been proved to hold in connection with the angles and the sides of a triangle. He will then be able to write down a sufficient number of equations by means of which the unknown quantities involved in the problem can be fully determined. We will consider only the simpler problems in which the objects lie all in one plane.

Suppose, for example, that an object P on the ground can be seen from two positions A and B. To calculate the distances of P from A and B, namely, PA and PB, we need only the measurement of the distance AB and of the angles PAB and PBA. For let

$$\hat{PAB} = \alpha, \hat{PBA} = \beta, \text{ and } AB = a;$$

$$\text{then } PA = \frac{a \sin \beta}{\sin (180^\circ - \alpha - \beta)} = \frac{a \sin \beta}{\sin (\alpha + \beta)},$$

$$\text{and } PB = \frac{a \sin \alpha}{\sin (180^\circ - \alpha - \beta)} = \frac{a \sin \alpha}{\sin (\alpha + \beta)}.$$

If, again, P be a balloon or any elevated object in the vertical plane through A and B, its distances from A and B and its height from the ground (PC) may be calculated as above from the measurement of AB and of the *angles of elevation* PAB and PBA.

$$\text{Thus } PC = PA \sin \alpha = \frac{a \sin \alpha \sin \beta}{\sin (\alpha + \beta)}.$$

149. We will next suppose that P and Q are two objects on a level plane, on which are also two positions A and B from which both P and Q can be seen. If we measure the distance AB and the angles PAB and PBA, we can determine, as in Art. 148, the distances PA and PB.

Similarly, from the measurement of AB and of the angles QAB and QBA, we may determine the distances QA and QB.

We may then obtain, by subtraction, the angles PAQ and PBQ. We may also verify these angles by direct measurement.

We may now calculate the distance PQ by using either the

measurements of AP, AQ and the angle PAQ, or those of BP, BQ and the angle PBQ. Thus the distance apart of two remote objects P and Q can be ascertained by measurements made at A and B, two positions far away perhaps from both of them.

The length AB which is subjected to actual measurement is called the **base line**; and the instrument which is used to measure the angle between two directions such as AB and AP is called a **Theodolite**.

150. A few simple examples of this class of problems were worked out in full in Art. 41; those which follow in the remainder of this section are somewhat more complicated. We will assume that the height of the observer's eye above the ground is taken account of in the given data, unless the contrary is expressly stated.

Example i. A man observes the angle of elevation of a pole to be 15° ; he walks directly towards it, in a horizontal plane passing through its base, for a distance c yards, and observes the angle of elevation to be now 75° . Find the height of the pole, and the distance of its base from the second place of observation.

Let AB be the pole; D and C respectively the first and second places of observation. Then

$$\hat{ADB} = 15^\circ; \hat{ACB} = 75^\circ; DC = c \text{ feet.}$$

Let $AB = h$ feet, and $CB = x$ feet.

$$\left. \begin{array}{l} \text{Then } \frac{x+c}{h} = \cot 15^\circ = 2 + \sqrt{3}, \\ \text{and } \frac{x}{h} = \cot 75^\circ = 2 - \sqrt{3}; \end{array} \right\}$$

$$\therefore \frac{c}{h} = 2\sqrt{3};$$

$$\therefore h = \frac{c}{2\sqrt{3}} = \frac{c}{6}\sqrt{3};$$

$$\text{and } x = (2 - \sqrt{3})h = \frac{c}{6}(2\sqrt{3} - 3).$$

Example ii. Find the angle subtended by a flagstaff 30 feet high, placed on the corner of a church tower 90 feet high, at a point on the ground 40 feet from the base of the corner.

Let AC be the flagstaff; CB , the tower; O , the place of observation; θ , the angle AOC .

Then $\tan \theta = \tan (AOB - COB)$

$$\begin{aligned} &= \frac{\frac{120}{40} - \frac{90}{40}}{1 + \frac{120}{40} \times \frac{90}{40}} \\ &= \frac{3}{31} \\ &= .0967742. \end{aligned}$$

In the usual way, Art. 111, by means of the Tables, we find that this value of the tangent belongs to an angle of $5^\circ 31' 39'' \cdot 14$;

$$\therefore \theta = 5^\circ 31' 39'' \cdot 14.$$

EXAMPLES 62.

$$[\sqrt{3} = 1.732; \sqrt{2} = 1.4142; \sqrt{6} = 2.449; \sqrt{115} = 10.7328.]$$

(1) The mast of a ship, 100 feet high, subtends an angle of 1° ; how far off is the ship? ($\cot 1^\circ = 57.289962$.)

(2) The angle of elevation of the top of a building is 24° , and a line drawn from the top to the point of observation is 60 yards long. Find the height of the building. ($\sin 24^\circ = .4067366$.)

(3) The vertical angle of an isosceles triangle is 120° , and the length of the perpendicular from the vertex on the base is 15 feet. Find the sides of the triangle.

(4) In a right-angled triangular field one side is 100 yards, and the perpendicular from the right angle on the hypotenuse is 50 yards. Find the other sides of the field.

(5) C is a post on the bank of a river; and along the opposite bank a man walks in a straight line from A to B , until the angle CAB is equal to the angle CBA . If each of these angles be 30° , and AB be 35 yards in length, find the width of the river.

(6) A person observes that the angle of elevation of the top of a tree is 60° ; he then walks to the foot of the tree and finds that his distance from it was 50 yards. Determine in feet and inches, to the nearest inch, the height of the tree.

(7) At the highest position of a rocket fired upwards on a dark night a luminous point was shown by the rocket and a weight was dropped from it. An observer found the angle of elevation of the luminous point to be $53^\circ 9'$, and the weight

reached the horizontal plane of observation at a distance 510 feet from the observer. Find approximately the vertical height to which the rocket had risen; given $\tan 53^\circ 9' = 1.334$.

(8) A bird, flying horizontally in a straight line from the top of a tree 65 feet high, is hit by a sportsman standing at the foot of the tree. It being known that the line of fire makes with the ground an angle of $40^\circ 12'$, find how far the bird had flown.

$$(\cot 40^\circ 12' = 1.1833402.)$$

(9) The string of a kite is 120 feet long and makes an angle of $37^\circ 30'$ with the ground. Find the height of the kite above the ground, and find how far from the holder a person must stand so that the kite may be vertically above his head.

$$(\sin 37^\circ 30' = .609, \cos 37^\circ 30' = .793.)$$

(10) The top of a spire 400 feet high, the base of which is a mile off, is just in a line with the top of a hill 1000 feet high. How far away horizontally is the summit of the hill?

(11) A ladder, placed at an angle of 75° with the ground, just reaches the sill of a window 27 feet above the ground on one side of a street. On turning the ladder over without moving its foot, it is found that when it rests against a wall on the other side of the street, it is at an angle of 15° with the ground. Find the breadth of the street.

(12) A person observes the angle of elevation of the top of a mountain to be 30° , and approaching 600 yards nearer he finds it to be 60° . Find the height of the mountain.

(13) A road runs in a straight line to the top of a hill. From the top of the hill two consecutive milestones are observed in the plane below, and their angles of depression are 30° and 45° respectively. Find the height of the hill.

(14) The direction of a lighthouse as seen from a ship makes an angle of 30° with the course of the ship. After the ship has advanced a mile on her course, the angle has increased to 60° . How near to the lighthouse will the ship pass?

(15) Two straight roads which cross one another meet a canal at angles of 30° and 60° respectively. If it be three miles, by the longer of the two roads, from the crossing to the canal, what is it by the shorter? And if there be a foot-path which goes the shortest way to the canal, what is the distance by it?

(16) A man standing upon a horizontal plane observes the

angle of elevation of the top of a tower to be 30° . He walks $100\sqrt{3}$ feet towards the tower, and then finds the angle of elevation to be 60° . Find the height of the tower, assuming that the observer's eye is 5 feet 6 inches from the ground.

(17) The elevation of the top of a tower to a person standing on the opposite bank of a river is 60° . When he has retired 100 feet in a straight line from the tower the elevation is 45° . Find the breadth of the river and the height of the tower, to the nearest foot.

(18) A ladder, c feet long, just reaches a window a feet high on one side of a street. When it is turned over without moving its foot it just reaches a window on the opposite side, and is then at an angle of 90° with its former position. Find the height of the latter window and the width of the street.

(19) A tower stands on a horizontal plane, and is observed from two points respectively 20 yards and 45 yards distant from its base. The angle of elevation of its summit in the former case is double that in the latter. Find the height of the tower.

(20) Find the tangent of the angle which a flagstaff 10 feet long, erected upon the summit of a tower 400 feet high, subtends at a point in a horizontal plane 200 feet from the base of the tower.

(21) A chimney 20 feet high, standing on the top of a building, subtends an angle whose tangent is $\frac{1}{6}$ at a distance of 70 feet from the foot of the building. Find the height of the building.

(22) A flagstaff 25 feet high is fixed on the top of a vertical bank, 15 feet high, which is close to the side of a road. At a point directly opposite, on the other side of the road, the flagstaff and the bank subtend equal angles. Find the width of the road.

(23) A tower 200 feet high stands on a level plane, and a turret 30 feet high is erected on the top of the tower. At a point on the plane the turret subtends the same angle as a man 6 feet high standing at the foot of the tower. Determine the distance of the tower from the observer. ($\sqrt{115} = 10.7238$.)

(24) A spire erected on the top of a tower 120 feet high, subtends an angle of 30° when viewed at a distance of 300 feet from the base of the tower. Find its height.

(25) A ladder CB is placed against a vertical wall AB , so that the angle ACB is 45° , and the angle CAB , 90° . It suddenly breaks, and rests upon the wall at a point D such that the angle DCB is 15° . If AC be 30 feet, find BD .

(26) I observe the elevation of the top of a tree to be 15° , and that of a taller tree 30 yards behind it to be 30° . I walk towards them until their tops are in a line, at a common elevation of 45° . Find the distance I walk.

(27) From a point A the elevation of the top B of a mound is 30° , and that of the top C of a cathedral beyond the mound is 60° . From B the elevation of C is 75° . If the cathedral be 347 feet high, find the distance of A from B.

(28) In ascending a slope inclined to the horizon at an angle of 15° , towards the foot of a tower, the elevations of the tower at stations 60 yards apart are observed to be 30° and 60° . Find the height of the tower.

(29) A tower subtends an angle α at a point on the same level as the foot of the tower, and at a second point, h feet above the first, the depression of the base of the tower is β . Find the height of the tower.

(30) At a point on a level with its base, the altitude of a tower is a , and a flagstaff of height c on the top of the tower subtends an angle β . Find the height of the tower.

XVI. CIRCLES AND OTHER FIGURES ASSOCIATED WITH A TRIANGLE.

(a) The circum-circle of a triangle.

151. DEFINITION. The circle which passes through the three angular points of a triangle is said to circumscribe it, and is called the **circumscribing-circle**.

For the sake of brevity, this circle is also called the **circum-circle**; its centre, the **circum-centre**; and its radius, the **circum-radius**.

The position of the circum-centre will always be denoted by the letter O, and the circum-radius by the symbol R.

152. *To find the radius of the circum-circle of a triangle.*

Let ABC be the triangle; O the circum-centre; BOH the diameter through B. Join CH.

Then, from Euclid III. 31, we know that the angle $BCH = 90^\circ$;
 $\therefore$ when A is acute, $BC = BH \sin BHC = BH \sin A$,

and when A is obtuse, the diameter BOH falling *below* BC,

$$BC = BH \sin BHC = BH \sin (180^\circ - A) = BH \sin A,$$

$\therefore$ in all cases, $a = 2R \sin A$,

$$\left. \begin{aligned} \therefore R &= \frac{a}{2 \sin A} \\ \text{Similarly, } R &= \frac{b}{2 \sin B} \\ \text{and } R &= \frac{c}{2 \sin C} \end{aligned} \right\}$$

From any of these expressions for R in terms of a side of the triangle and the opposite angle, we may deduce an expression in terms of the three sides and the area of the triangle. Thus

$$R = \frac{a}{2 \sin A} = \frac{abc}{2bc \sin A} = \frac{abc}{4\Delta}.$$

NOTE I. The three expressions for R , in terms of a side of the triangle and the opposite angle, may be written in the form

$$a = 2R \sin A, \quad b = 2R \sin B, \quad c = 2R \sin C.$$

These forms are very useful in dealing with transformations and identities.

NOTE II. It will be seen that **any chord** of a circle is equal to the **diameter** of the circle multiplied by the **sine of the angle in the segment cut off by the chord**.

(b) The in-circle of a triangle.

153. DEFINITION. The circle which touches each of the three sides is called the **inscribed circle** of the triangle.

For brevity, it is also called the **in-circle**; its radius, the **in-radius**; and its centre, the **in-centre**.

The position of the in-centre will always be denoted by the letter I , and the in-radius by the symbol r .

154. *To find the radius of the in-circle of a triangle.*

Let ABC be the triangle; I its in-centre. We are told in Euclid iv. 4 how the in-circle can be drawn.

Join AI , BI , CI . Then we know that these lines bisect the angles A , B , and C respectively; and, if IH , IK , and IL be drawn perpendicular to the sides, meeting them respectively at the points H , K , and L , we know that H , K , and L are the points at which the in-circle touches the sides.

- (i) We have $\triangle ABC = \triangle BIC + \triangle CIA + \triangle AIB$,
 $\therefore 2\Delta = ra + rb + rc = r(a + b + c) = 2rs$;
 $\therefore r = \frac{\Delta}{s}$.

(ii) Again, we have

$$\begin{aligned} a &= BC \\ &= BH + HC \\ &= HI \cot HBI + HI \cot HCI \\ &= r \cot \frac{B}{2} + r \cot \frac{C}{2} \\ &= r \left(\cot \frac{B}{2} + \cot \frac{C}{2} \right); \end{aligned}$$

$$\begin{aligned} \therefore r &= \frac{a}{\cot \frac{B}{2} + \cot \frac{C}{2}} = \frac{a \sin \frac{B}{2} \sin \frac{C}{2}}{\cos \frac{A}{2}}. \\ \text{Similarly, } r &= \frac{b}{\cot \frac{C}{2} + \cot \frac{A}{2}} = \frac{b \sin \frac{C}{2} \sin \frac{A}{2}}{\cos \frac{B}{2}}, \\ \text{and } r &= \frac{c}{\cot \frac{A}{2} + \cot \frac{B}{2}} = \frac{c \sin \frac{A}{2} \sin \frac{B}{2}}{\cos \frac{C}{2}}. \end{aligned}$$

Each of these expressions for r may be reduced to

$$4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

(c) The escribed circles of a triangle.

155. DEFINITION. A circle which touches any one side of a triangle, and the other two sides *produced*, is called an **escribed circle** of the triangle.

There are three escribed circles belonging to every triangle:—

- (i) that which touches BC between B and C , and also touches AB and AC produced; (ii) that which touches AC between A and

C, and also touches BA and BC produced; (iii) that which touches AB between A and B, and also touches CA and CB produced.

The positions of the three centres will be denoted by the letters I_1 , I_2 , and I_3 respectively; and the radii respectively by the symbols r_1 , r_2 and r_3 .

156. *To find the radii of the escribed circles of a triangle.*

Let ABC be a triangle, having the side AB produced to X and AC to Y.

The centre of a circle touching BC and BX must lie on the bisector of $\hat{CBX}$; and the centre of a circle touching BC and CY must lie on the bisector of $\hat{BCY}$. Hence BI_1 and CI_1 bisect $\hat{CBX}$ and $\hat{BCY}$ respectively.

Also AI_1 bisects $\hat{BAC}$, and therefore passes through the in-centre I.

Draw I_1L_1 , I_1M_1 , I_1N_1 respectively perpendicular to BC, CY and BX. Then L_1 , M_1 and N_1 are the points at which this escribed circle touches these lines, and the three perpendiculars are each equal to r_1 .

$$\begin{aligned} \text{(i) We have } \triangle ABC &= \triangle AI_1B + \triangle AI_1C - \triangle BI_1C, \\ \therefore 2\Delta &= r_1c + r_1b - r_1a = r_1(c + b - a) = r_1(2s - 2a), \\ \therefore \Delta &= r_1(s - a), \\ \therefore r_1 &= \frac{\Delta}{s - a}. \end{aligned}$$

$$\text{Similarly, } r_2 = \frac{\Delta}{s - b}, \text{ and } r_3 = \frac{\Delta}{s - c}.$$

(ii) Again, we have

$$\begin{aligned} a &= BC = BL_1 + L_1C \\ &= I_1L_1 \cot CBI_1 + I_1L_1 \cot BCI_1 \\ &= r_1 \cot \frac{1}{2}(180^\circ - B) + r_1 \cot \frac{1}{2}(180^\circ - C) \\ &= r_1 \left(\tan \frac{B}{2} + \tan \frac{C}{2} \right); \end{aligned}$$

$$\therefore r_1 = \frac{a}{\tan \frac{B}{2} + \tan \frac{C}{2}} = \frac{a \cos \frac{B}{2} \cos \frac{C}{2}}{\cos \frac{A}{2}} = 4R \sin \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

Similarly,

$$r_2 = \frac{b}{\tan \frac{C}{2} + \tan \frac{A}{2}} = \frac{b \cos \frac{C}{2} \cos \frac{A}{2}}{\cos \frac{B}{2}} = 4R \cos \frac{A}{2} \sin \frac{B}{2} \cos \frac{C}{2},$$

and

$$r_3 = \frac{c}{\tan \frac{A}{2} + \tan \frac{B}{2}} = \frac{c \cos \frac{A}{2} \cos \frac{B}{2}}{\cos \frac{C}{2}} = 4R \cos \frac{A}{2} \cos \frac{B}{2} \sin \frac{C}{2}.$$

EXAMPLES 63.

Find the radius of the circum-circle of a triangle when one side and the opposite angle are respectively

(1) 5 ft., 90° .	(2) 10 in., 30° .
(3) 6 ft., 45° .	(4) 18 ft., 60° .
(5) 20 ft., 150° .	(6) 34 ft., 15° .

Find the area, the circum-radius, the in-radius and the radii of the escribed circles, of a triangle whose sides are

(7) 3, 4, 5.	(8) 10, 12, 10.
(9) 3, 3, 4.	(10) 10, 15, 20.

Find r , r_1 , r_2 and r_3 , when

(11) $b=16$, $A=120^\circ$, $C=30^\circ$.	(12) $a=20$, $B=30^\circ$, $C=60^\circ$.
(13) $c=15$, $B=90^\circ$, $A=45^\circ$.	(14) $R=15$, $C=60^\circ$, $A=60^\circ$.

157. The lengths, p_1 , p_2 and p_3 , of the perpendiculars from the angular points to the opposite sides, are connected respectively with the lengths a , b and c of those sides by the relations

$$ap_1 = 2\Delta, \quad bp_2 = 2\Delta \quad \text{and} \quad cp_3 = 2\Delta;$$

hence
$$a = \frac{2\Delta}{p_1}, \quad b = \frac{2\Delta}{p_2} \quad \text{and} \quad c = \frac{2\Delta}{p_3}.$$

EXAMPLES 64.

Prove that

(1) $\Delta = \sqrt{rr_1r_2r_3}.$	(2) $\frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} = \frac{1}{r}.$
(3) $\frac{1}{p_1} + \frac{1}{p_2} - \frac{1}{p_3} = \frac{1}{r_3}.$	(4) $\frac{1}{2Rr} = \frac{1}{ab} + \frac{1}{ac} + \frac{1}{bc}.$

$$(5) \quad r = (s - a) \tan \frac{A}{2}.$$

$$(6) \quad r_1 = s \tan \frac{A}{2}, \quad r_2 = s \tan \frac{B}{2}, \quad r_3 = s \tan \frac{C}{2}.$$

$$(7) \quad r_1 r_2 + r_2 r_3 + r_3 r_1 = \frac{1}{4} (a + b + c)^2.$$

$$(8) \quad 4r_1 r_2 r_3 = r (a + b + c)^2.$$

$$(9) \quad \frac{a}{r_1 (r_2 + r_3)} = \frac{b}{r_2 (r_3 + r_1)} = \frac{c}{r_3 (r_1 + r_2)} = \frac{\Delta}{r_1 r_2 r_3}.$$

$$(10) \quad r_1 r_2 = ab \cos^2 \frac{C}{2}. \quad (11) \quad \frac{r}{\sqrt{(r_2 - r)(r_3 - r)}} = \sin \frac{A}{2}.$$

$$(12) \quad r_1 \cot \frac{A}{2} = r_2 \cot \frac{B}{2} = r_3 \cot \frac{C}{2}.$$

$$(13) \quad r_2 r_3 \tan^2 \frac{A}{2} = r r_1.$$

$$(14) \quad r_1 r_2 r_3 \tan^2 \frac{A}{2} \tan^2 \frac{B}{2} \tan^2 \frac{C}{2} = r^3.$$

$$(15) \quad (r_3 - r) \cot \frac{C}{2} = (r_1 + r_2) \tan \frac{C}{2} = c.$$

$$(16) \quad 8R \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = a + b + c.$$

$$(17) \quad 4R \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} = r.$$

$$(18) \quad (r_1 - r)(r_2 - r)(r_3 - r) = 4r^2 R = \frac{abc r^2}{\Delta}.$$

$$(19) \quad R r_1 (\sin B + \sin C - \sin A) = \Delta.$$

$$(20) \quad \frac{1}{r_1^2} + \frac{1}{r_2^2} + \frac{1}{r_3^2} + \frac{1}{r^2} = \frac{a^2 + b^2 + c^2}{\Delta^2}.$$

$$(21) \quad \frac{(r_2 + r_3)(r_3 + r_1)(r_1 + r_2)}{r_2 r_3 + r_3 r_1 + r_1 r_2} = 4R.$$

$$(22) \quad a \operatorname{cosec}^2 \frac{A}{2} + b \operatorname{cosec}^2 \frac{B}{2} + c \operatorname{cosec}^2 \frac{C}{2} = \frac{abc}{r^2}.$$

$$(23) \quad \text{If } r : r_1 = r_2 : r_3, \text{ prove that } C = 90^\circ.$$

(24) In the ambiguous case, prove that the two triangles have equal circum-circles.

158. To find the segments of the sides of a triangle made by the points of contact of the inscribed and the escribed circles, in terms of a , b , c and s .

We will use the notation of the preceding articles.

(i) Since tangents drawn to a circle from an external point are equal, we have

$$AN = AM, \quad BN = BL, \quad \text{and} \quad CM = CL;$$

$$\therefore 2(AM + BL + CL) = AB + BC + AC \\ = 2s,$$

$$\therefore AM + BL + CL = s,$$

$$\therefore AM + a = s,$$

$$\therefore AM = s - a = AN.$$

$$\left. \begin{array}{l} \text{Similarly,} \quad BL = s - b = BN, \\ \text{and} \quad \quad \quad CL = s - c = CM. \end{array} \right\}$$

(ii) Again, we have

$$BN_1 = BL_1, \quad CM_1 = CL_1, \quad AN_1 = AM_1;$$

$$\therefore AN_1 + AM_1 = AB + BL_1 + AC + CL_1 \\ = AB + BC + AC \\ = 2s,$$

$$\therefore AN_1 = s = AM_1.$$

$$\left. \begin{array}{l} \text{Also} \quad \quad \quad BN_1 = s - c = BL_1, \\ \text{and} \quad \quad \quad CM_1 = s - b = CL_1. \end{array} \right\}$$

(iii) From the results (i) and (ii) many others may be deduced. For example,

$$MM_1 = AM_1 - AM = s - (s - a) = a,$$

$$\text{and} \quad NN_1 = AN_1 - AN = s - (s - a) = a,$$

$$\therefore MM_1 = NN_1 = a.$$

$$\text{Similarly,} \quad LL_1 = BL_1 - BL = (s - c) - (s - b) = b - c.$$

NOTE. The student should draw the other two escribed circles, and should write down the lengths of the lines which are situated similarly to those considered above.

159. DEFINITION. If from each angular point of a triangle a straight line be drawn perpendicular to the opposite side, and a triangle be formed by joining the *feet* of these perpendiculars, the triangle so formed is called the **Pedal Triangle**.

The three perpendiculars meet at a point which is called the **Orthocentre** of the original triangle. Its position will always be denoted by the letter P.

160. *To find the sides and angles of the Pedal Triangle.*

Let ABC be the original triangle, and XYZ the pedal triangle. Then since $\hat{BZC}$ and $\hat{BYC}$ are right angles, it follows that the figure BZYC is cyclic;

$\therefore \hat{AYZ} = 180^\circ - \hat{ZYC} = B$,
and $\hat{AZY} = 180^\circ - \hat{BZY} = C$;

$\therefore \triangle AZY$ is similar to $\triangle ACB$;

$\therefore$ corresponding sides are proportional.

But $AY = AB \cos A$, and $AZ = AC \cos A$;

$\therefore ZY = BC \cos A$,

i.e., $ZY = a \cos A = R \sin 2A$.

Similarly,

$XZ = b \cos B = R \sin 2B$,

and

$YX = c \cos C = R \sin 2C$.

Also

$\hat{ZXA} = 90^\circ - \hat{ZXB} = 90^\circ - A$,

and

$\hat{YXA} = 90^\circ - \hat{YXC} = 90^\circ - A$;

$\therefore \hat{ZXY} = 180^\circ - 2A$.

Similarly,

$\hat{XZY} = 180^\circ - 2B$,

and

$\hat{XZY} = 180^\circ - 2C$.

Thus the sides of the pedal triangle are $a \cos A$, $b \cos B$ and $c \cos C$, or $R \sin 2A$, $R \sin 2B$, and $R \sin 2C$; and the angles respectively opposite to them are

$180^\circ - 2A$, $180^\circ - 2B$ and $180^\circ - 2C$.

NOTE. BC being the diameter of the circle described about the quadrilateral BZYC, it follows at once from Art. 152, Note II, that $YZ = BC \sin ZCY = a \cos A$.

NOTE A. Since $\hat{ZXA}$ and $\hat{YXA}$ are each equal to $90^\circ - A$, we see that XA bisects $\hat{ZXY}$. Similarly YB and ZC respectively bisect $\hat{XYZ}$ and $\hat{YZX}$. Hence P , the orthocentre of the original triangle, is the incentre of the pedal triangle.

NOTE B. The circum-radius of the pedal triangle

$$= \frac{YZ}{2 \sin YXZ} = \frac{a \cos A}{2 \sin (180^\circ - 2A)} = \frac{a \cos A}{2 \sin 2A} = \frac{a}{4 \sin A} = \frac{R}{2}.$$

Also the circum-circle of the pedal triangle is the **Nine Point Circle** of the original triangle; hence the radius of the nine-point circle of the triangle ABC is $\frac{1}{2}R$.

NOTE C. It may be proved also that the escribed circles of the triangle XYZ have their centres respectively at A , B and C .

EXAMPLES 65.

- (1) Prove that the distances l_1, l_2, l_3 are respectively

$$a \sec \frac{A}{2}, \quad b \sec \frac{B}{2}, \quad c \sec \frac{C}{2}.$$

- (2) Prove that the distances l_2l_3, l_3l_1, l_1l_2 are respectively

$$a \operatorname{cosec} \frac{A}{2}, \quad b \operatorname{cosec} \frac{B}{2}, \quad c \operatorname{cosec} \frac{C}{2}.$$

- (3) Prove that $AP = 2R \cos A$, and write down the corresponding values of BP and CP .

- (4) Prove that $XP = 2R \cos B \cos C$ and write down the corresponding values of YP and ZP .

- (5) Prove that $AI = r \operatorname{cosec} \frac{A}{2} = 4R \sin \frac{B}{2} \sin \frac{C}{2}$.

- (6) Prove that $AI_1 = r_1 \operatorname{cosec} \frac{A}{2} = 4R \cos \frac{B}{2} \cos \frac{C}{2}$.

- (7) If the bisectors of the angles A, B, C of a triangle meet the opposite sides at Q, R, S respectively, prove that

$$BQ = \frac{ac}{b+c}; \quad CQ = \frac{ab}{b+c}.$$

- (8) If, in a triangle ABC , the perpendicular from A on BC meets BC in D and the circum-circle in E , prove that

$$AE = 2R \cos (B - C), \quad \text{and} \quad DE = 2R \cos B \cos C.$$

PART IV.

XVII. TRIGONOMETRICAL EQUATIONS.

161. For any stated angle, each of the trigonometrical ratios has a definite numerical value and a definite algebraic sign; but, on the other hand, a stated value of a trigonometrical ratio does not in any case belong to one, and only one, definite angle. In fact, even if the complete values of *all* the trigonometrical ratios be given, we cannot say more than that the angle to which they belong must be *one* of a certain set.

We know, for example, that the sine of 45° is $\frac{1}{\sqrt{2}}$, and cannot have any other value; but, if the value of a sine is stated to be $\frac{1}{\sqrt{2}}$, it does not follow that the angle is necessarily 45° . It may be 45° or 135° , even if we consider such angles only as are less than four right angles; while, if the magnitude be *unrestricted*, the angle may be any positive angle belonging to either of the sets

$45^\circ, 360^\circ + 45^\circ, 720^\circ + 45^\circ, \dots \text{ad infinitum},$

and $135^\circ, 360^\circ + 135^\circ, 720^\circ + 135^\circ, \dots \text{ad infinitum};$

or it may be any negative angle belonging to either of two similar sets. Thus when there is no limit as to *magnitude*, there is also no limit to the *number* of angles which have $\frac{1}{\sqrt{2}}$ for sine.

We may say, briefly, that if the complete value of any one, but only one, ratio be given, there are *two* of the angles to which it belongs in every revolution; for example, in the first revolution, there are two angles, namely, 30° and 150° , which have $+\frac{1}{2}$ for their sine. If, however, the complete values of any *two* ratios be given, one of which is not the reciprocal of the other, there is *one* angle

only in each revolution to which they both can belong; 30° , for example, being the only angle less than four right angles which has both $\frac{1}{2}$ for its sine and $\frac{1}{\sqrt{3}}$ for its tangent.

It is desirable that we should be able to represent the infinite series of angles to which a particular value of a trigonometrical ratio may belong by means of concise formulæ. In the diagrams of the following articles, we assume that one of the angles which have the given ratio is in each case less than a right angle. This assumption simplifies the diagram, and does not affect the method of proof.

Formula for all angles which have a given tangent or cotangent.

162. We will suppose the angles to be generated by a straight line starting from the position OX , and turning about the point O . Let a be the circular measure of the least positive angle which has the given tangent or cotangent; and let $\angle XO A = a$, and $\angle XO A' = \pi + a$.

Then $\angle XO A'$ is clearly the only other angle in the first revolution which has the same tangent or cotangent as a . Hence we have to find an expression which shall represent *all* angles for which the rotating line, starting from the position OX , stops in either of the positions OA or OA' .

It will be seen from the figure that the *positive* angles to be represented are the angles of the series

$$a, \pi + a, 2\pi + a, 3\pi + a, 4\pi + a, 5\pi + a, \dots \&c.;$$

and that the *negative* angles are the angles of the series

$$-\pi + a, -2\pi + a, -3\pi + a, -4\pi + a, -5\pi + a, \dots \&c.$$

All these angles, and no others, form a group which may be represented by $n\pi + a$, where n stands for zero or any positive or negative *integer*, and a is the circular measure of the least positive angle which has the given tangent or cotangent.

NOTE. We have said that the symbol a represents the *least positive* angle which has the given tangent or cotangent. This supposition simplifies the figure, but *theoretically* it is not necessary. In applying the formula to particular cases, it will however

be best for the beginner at first to assume that a has this meaning. The least positive angle which has a given ratio we will call the **prime** angle of the series. All the angles which belong to the group represented by $n\pi + a$ may be obtained by giving to n the values 0, 1, 2, 3, 4,..... and $-1, -2, -3, -4, \dots$. The number of the angles so obtained is infinitely great.

The expression $n\pi + a$ is called the **general value** of the angles which have the stated tangent or cotangent, as distinguished from the *particular* angles obtained from it by giving *particular* values to n .

163. We will apply the formula obtained in Art. 162 to a few simple cases.

Example i. Find a single expression which shall represent all the angles whose tangent is $\frac{1}{\sqrt{3}}$; i.e., solve the equation $\tan \theta = \frac{1}{\sqrt{3}}$.

We have
$$\tan \theta = \frac{1}{\sqrt{3}}.$$

The prime angle $= \frac{\pi}{6}$; $\therefore \theta = n\pi + \frac{\pi}{6}.$

Example ii. Solve the equation $\tan \theta = -\frac{1}{\sqrt{3}}$.

We have
$$\tan \theta = -\frac{1}{\sqrt{3}}.$$

The prime angle $= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$; $\therefore \theta = n\pi + \frac{5\pi}{6}.$

Example iii. Solve the equation $\cot^2 \theta = 3$.

We have
$$\cot^2 \theta = 3;$$

$$\therefore \tan^2 \theta = \frac{1}{3};$$

$$\therefore \tan \theta = \pm \frac{1}{\sqrt{3}}.$$

The prime angles are $\frac{\pi}{6}$ and $\pi - \frac{\pi}{6}$, i.e., $\frac{\pi}{6}$ and $\frac{5\pi}{6}$;

$$\therefore \theta = n\pi + \frac{\pi}{6} \text{ or } n\pi + \frac{5\pi}{6}.$$

These results may be put together in the form $n\pi \pm \frac{\pi}{6}.$

EXAMPLES 66.

Write down the general values of θ in the following cases:—

(1) $\tan \theta = \sqrt{3}.$

(2) $\tan \theta = -\sqrt{3}.$

(3) $\tan \theta = 1.$

(4) $\tan \theta = -1.$

(5) $\tan \theta = 0.$

(6) $\cot \theta = \sqrt{3}.$

(7) $\cot \theta = -\sqrt{3}.$

(8) $\cot \theta = -1.$

(9) $\tan^2 \theta = 1.$

(10) $\cot^2 \theta = \frac{1}{3}.$

Formula for all angles which have a given cosine or secant.

164. We will suppose the angles to be generated by a straight line starting from the position OX , and turning about the point O . Let α be the circular measure of the least positive angle which has the given cosine or secant; and let $\angle XOA = \alpha$, and $\angle XOA' = 2\pi - \alpha$.

The angle $2\pi - \alpha$ is clearly the only other angle in the first revolution which has the same cosine or secant as α . Hence we have to find an expression which shall represent *all* angles for which the rotating line, starting from the position OX , stops in either of the positions OA or OA' .

It will be seen from the figure that the *positive* angles to be represented are the angles of the series

$$\alpha, 2\pi - \alpha, 2\pi + \alpha, 4\pi - \alpha, 4\pi + \alpha, 6\pi - \alpha, 6\pi + \alpha, \dots \&c.;$$

and that the *negative* angles are the angles of the series

$$-\alpha, -2\pi + \alpha, -2\pi - \alpha, -4\pi + \alpha, -4\pi - \alpha, -6\pi + \alpha, -6\pi - \alpha, \dots \&c.$$

All these angles, and no others, form a group which may be represented by $2n\pi \pm \alpha$, where n stands for zero or any positive or negative *integer*, and α is the circular measure of the least positive angle which has the given cosine or secant.

NOTE. If, in the *general* formula $2n\pi \pm \alpha$, we substitute for n the numbers 0, 1, 2, 3,, -1, -2, -3, in succession, we obtain the *particular* angles which it represents.

Example i. Solve the equation $\cos \theta = \frac{1}{\sqrt{2}}$.

We have $\cos \theta = \frac{1}{\sqrt{2}}$.

The prime angle $= \frac{\pi}{4}$; $\therefore \theta = 2n\pi \pm \frac{\pi}{4}$.

Example ii. Solve the equation $\sec \theta = -\sqrt{2}$.

We have $\sec \theta = -\sqrt{2}$;

$$\therefore \cos \theta = -\frac{1}{\sqrt{2}}.$$

The prime angle $= \pi - \frac{\pi}{4} = \frac{3\pi}{4}$; $\therefore \theta = 2n\pi \pm \frac{3\pi}{4}$.

EXAMPLES 67.

Write down the general values of θ in the following cases:—

(1) $\cos \theta = \frac{1}{2}$.

(2) $\sec \theta = -2$.

(3) $\cos \theta = \frac{\sqrt{3}}{2}$.

(4) $\sec \theta = -\frac{2}{\sqrt{3}}$.

(5) $\cos \theta = 1$.

(6) $\sec \theta = -1$.

(7) $\cos \theta = 0$.

(8) $\sec \theta = 1$.

(9) $\cos^2 \theta = \frac{3}{4}$.

(10) $\cos^2 \theta = \frac{1}{4}$.

Formula for all angles which have a given sine or cosecant.

165. We will suppose the angles to be generated by a straight line starting from the position OX , and turning about the point O . Let α be the circular measure of the least positive angle which has the given sine or cosecant; and let $\angle XOA = \alpha$, and $\angle XOA' = \pi - \alpha$.

The angle $\pi - \alpha$ is clearly the only other angle in the first revolution which has the same sine or cosecant as α . Hence we have to find an expression which shall represent all angles for which the rotating line, starting from the position OX , stops in either of the positions OA or OA' .

It will be seen from the figure that the *positive* angles are the angles of the series

$$a, \pi - a, 2\pi + a, 3\pi - a, 4\pi + a, 5\pi - a, \dots \&c. ;$$

and that the *negative* angles are the angles of the series

$$-\pi - a, -2\pi + a, -3\pi - a, -4\pi + a, -5\pi - a, \dots \&c.$$

All these angles, and no others, may be put into one or other of the groups represented by

$$(2n + 1)\pi - a \quad \text{and} \quad 2n\pi + a,$$

where $2n + 1$ stands for any *odd* integer, and $2n$ for any *even* integer, positive or negative. And these two groups may be combined into the single group represented by $n\pi + (-1)^n a$, where n is zero or *any* positive or negative integer, and a is the least positive angle which has the given sine or cosecant.

NOTE. The student should realise clearly that $n\pi + (-1)^n a$ does really represent both of the expressions $2n\pi + a$ and $(2n + 1)\pi - a$, as stated above. It will help him to do so, if in the expression $n\pi + (-1)^n a$ he substitutes for n successively

an *even* integer, positive or negative,

and an *odd* integer, positive or negative.

The results will be

(an *even* multiple of π) + a , and (an *odd* multiple of π) - a ;
which correspond respectively to

$$2n\pi + a \quad \text{and} \quad (2n + 1)\pi - a.$$

Example i. Solve the equation $\sin \theta = \frac{\sqrt{3}}{2}$.

We have $\sin \theta = \frac{\sqrt{3}}{2}$.

$$\text{The prime angle} = \frac{\pi}{3} ; \therefore \theta = n\pi + (-1)^n \frac{\pi}{3}.$$

Example ii. Solve the equation $\sin \theta = -\frac{\sqrt{3}}{2}$.

We have $\sin \theta = -\frac{\sqrt{3}}{2}$.

$$\text{The prime angle} = \pi + \frac{\pi}{3} = \frac{4\pi}{3} ; \therefore \theta = n\pi + (-1)^n \frac{4\pi}{3}.$$

EXAMPLES 68.

Write down the general value of θ , when

- (1) $\sin \theta = \frac{1}{2}$. (2) $\sin \theta = -\frac{1}{2}$. (3) $\sin \theta = \frac{1}{\sqrt{2}}$.
 (4) $\operatorname{cosec} \theta = -\sqrt{2}$. (5) $\sin \theta = 0$. (6) $\sin \theta = 1$.
 (7) $\operatorname{cosec} \theta = -1$. (8) $\sin^2 \theta = \frac{1}{4}$. (9) $\sin^2 \theta = 1$.

166. From the equation $\sin^2 \theta = \frac{1}{3}$, we see that

$$\sin \theta = +\frac{1}{\sqrt{3}}, \text{ or } \sin \theta = -\frac{1}{\sqrt{3}}.$$

These values lead to four positions of the revolving line, as in the accompanying figure. There is one position in each quadrant and they are symmetrical; and all the angles corresponding to the four positions may clearly be grouped in the form

$$n\pi \pm \frac{\pi}{6}.$$

The *general* form of the group is $n\pi \pm a$, where a is the prime angle which has the given sine.

It will be found also that the same formula $n\pi \pm a$ represents the solutions of such equations also as $\cos^2 \theta = x$ and $\tan^2 \theta = x$, as well as that of $\sin^2 \theta = x$.

167. If the complete values of *two* ratios are given the revolving line can have only *one* position (Art. 161). Consequently, if a be the *prime* angle for which the two ratios have the given values, it follows that *all* the angles whose ratios have these values will form a group which may be represented by $2n\pi + a$, where n is zero or any positive or negative integer.

Example. Find the complete value of θ which satisfies the equations

$$\sin \theta = -\frac{1}{2}, \quad \cos \theta = -\frac{\sqrt{3}}{2}.$$

$$\text{The prime angle} = \pi + \frac{\pi}{6} = \frac{7\pi}{6}; \quad \therefore \theta = 2n\pi + \frac{7\pi}{6}.$$

EXAMPLES 69.

Write down the general value of θ in the following cases:—

- (1) $\tan^2 \theta = 1$. (2) $\cot^2 \theta = \frac{1}{3}$.
 (3) $\cos^2 \theta = \frac{1}{4}$. (4) $\cos^2 \theta = \frac{3}{4}$.
 (5) $\sin^2 \theta = 1$. (6) $\operatorname{cosec}^2 \theta = 2$.

Shew the position of the revolving line, and write down the general value of θ , in the following cases:—

- (7) $\sin \theta = \frac{1}{2}$, $\cos \theta = \frac{\sqrt{3}}{2}$. (8) $\cos \theta = \frac{1}{2}$, $\sin \theta = -\frac{\sqrt{3}}{2}$.
 (9) $\cos \theta = \frac{1}{\sqrt{2}}$, $\tan \theta = 1$. (10) $\sin \theta = -\frac{1}{\sqrt{2}}$, $\tan \theta = -1$.
 (11) $\cos \theta = -\frac{1}{2}$, $\tan \theta = \sqrt{3}$. (12) $\sin \theta = -\frac{\sqrt{3}}{2}$, $\cos \theta = -\frac{1}{2}$.

Miscellaneous Equations.

168. In the case of an equation of the form

$$a \sin \theta + b \cos \theta = c,$$

the artifice used in the following example should be noticed.

Example. Solve the equation $\sqrt{3} \sin \theta - \cos \theta = 1$.

We have $\sqrt{3} \sin \theta - \cos \theta = 1$;

$$\therefore \frac{1}{2} \cos \theta - \frac{\sqrt{3}}{2} \sin \theta = -\frac{1}{2};$$

$$\therefore \cos \theta \cos \frac{\pi}{3} - \sin \theta \sin \frac{\pi}{3} = -\frac{1}{2};$$

$$\therefore \cos \left(\theta + \frac{\pi}{3} \right) = -\frac{1}{2};$$

$$\therefore \theta + \frac{\pi}{3} = 2n\pi \pm \frac{2\pi}{3};$$

$$\therefore \theta = 2n\pi - \frac{\pi}{3} \pm \frac{2\pi}{3};$$

$$\therefore \theta = 2n\pi + \frac{\pi}{3}, \text{ or } (2n-1)\pi.$$

EXAMPLES 70.

Solve the equations :—

$$(1) \quad \sin \theta + \cos \theta = 1.$$

$$(2) \quad \sin \theta - \cos \theta = 1.$$

$$(3) \quad \sin \theta - \cos \theta = -1.$$

$$(4) \quad \sin \theta + \sqrt{3} \cos \theta = 1.$$

$$(5) \quad \sin \theta - \sqrt{3} \cos \theta = 1.$$

$$(6) \quad \sqrt{3} \sin \theta - \cos \theta = -1.$$

$$(7) \quad \sin \theta + \cos \theta = \sqrt{2}.$$

$$(8) \quad \sin \theta - \cos \theta = -\sqrt{2}.$$

$$(9) \quad \sin \theta - \cos \theta = -\sqrt{\frac{3}{2}}.$$

169. We will now consider methods of solving trigonometrical equations of a more general kind. In the various processes which may be adopted, the student will find employment for the skill which he should now have acquired in the use and manipulation of trigonometrical formulæ.

Example i. Solve the equation $\sin 2x = \sin x$.

1ST SOLUTION.

$$\sin 2x = \sin x,$$

$$\therefore \sin 2x - \sin x = 0,$$

$$\therefore 2 \cos \frac{3x}{2} \sin \frac{x}{2} = 0,$$

$$\therefore \cos \frac{3x}{2} = 0, \text{ or } \sin \frac{x}{2} = 0.$$

The prime angles are $\frac{\pi}{2}$ and 0, respectively ;

$$\therefore \frac{3x}{2} = 2n\pi \pm \frac{\pi}{2}, \text{ or } \frac{x}{2} = n\pi ;$$

$$\therefore x = \frac{2}{3} \left(2n\pi \pm \frac{\pi}{2} \right), \text{ or } 2n\pi.$$

2ND SOLUTION.

$$\sin 2x = \sin x,$$

$$\therefore 2 \sin x \cos x - \sin x = 0,$$

$$\therefore \sin x (2 \cos x - 1) = 0,$$

$$\therefore \sin x = 0, \text{ or } 2 \cos x - 1 = 0,$$

$$\therefore \sin x = 0, \text{ or } \cos x = \frac{1}{2}.$$

The prime angles are 0 and $\frac{\pi}{3}$, respectively ;

$$\therefore x = n\pi, \text{ or } 2n\pi \pm \frac{\pi}{3}.$$

NOTE A. In the 1st solution it must be noticed that we obtain the *general* values of $\frac{3x}{2}$ and $\frac{x}{2}$, and that we deduce the general values of x from these.

NOTE B. The results obtained by the second process may seem to disagree with those obtained by the former. If however the values, 0, +1, -1, +2, -2, &c., be substituted for n , it will be found that both *general* solutions lead to identically the same *particular* solutions, but not necessarily in the same order. A trigonometrical equation may often be solved in more than one

way, but the different *general* solutions will **always** be found to yield the same series of *particular* angles.

Example ii. Solve the equation $\sin 2\theta - \cos 2\theta = \cos \theta - \sin \theta$.

We have $\sin 2\theta - \cos 2\theta = \cos \theta - \sin \theta$;
 $\therefore$ transposing, $\sin 2\theta + \sin \theta = \cos 2\theta + \cos \theta$,
 $\therefore 2 \sin \frac{3\theta}{2} \cos \frac{\theta}{2} = 2 \cos \frac{3\theta}{2} \cos \frac{\theta}{2}$,
 $\therefore \cos \frac{\theta}{2} \left(\sin \frac{3\theta}{2} - \cos \frac{3\theta}{2} \right) = 0$,
 $\therefore \cos \frac{\theta}{2} = 0$, or $\sin \frac{3\theta}{2} - \cos \frac{3\theta}{2} = 0$,
 $\therefore \cos \frac{\theta}{2} = 0$, or $\tan \frac{3\theta}{2} = 1$,
 $\therefore \frac{\theta}{2} = 2n\pi \pm \frac{\pi}{2}$, or $\frac{3\theta}{2} = n\pi + \frac{\pi}{4}$,
 $\therefore \theta = 4n\pi \pm \pi$, or $\frac{2}{3} \left(n\pi + \frac{\pi}{4} \right)$.

No definite rules can be given for the preliminary treatment of a trigonometrical equation. The student should practise with a great many, and at first these should be easy. Experience alone can guide him as to the best method to be adopted.

EXAMPLES 71.

Solve the equations:—

(1) $\sin 3\theta = \frac{1}{2}$.	(2) $\cos 2\theta = \frac{1}{\sqrt{2}}$.	(3) $\cos 5\theta = -1$.
(4) $\tan 6\theta = -\frac{1}{\sqrt{3}}$.	(5) $\sin \frac{\theta}{2} = -\frac{1}{\sqrt{2}}$.	(6) $\sec^2 \frac{\theta}{4} = \frac{4}{3}$.
(7) $\cot^2 2\theta = 3$.	(8) $\sin^2 \frac{3\theta}{2} = \frac{3}{4}$.	
(9) $\sin \theta + \operatorname{cosec} \theta = 2$.	(10) $\sin \theta - \cos \theta = 0$.	
(11) $\cos^2 \theta = \sin^2 \theta$.	(12) $\sin \theta = \tan \theta$.	
(13) $2 \cos x = \sqrt{3} \cot x$.	(14) $\tan \theta + \cot \theta = 2$.	
(15) $\sec \theta = \frac{3}{2} + \cos \theta$.		
(16) $\tan^2 \theta - (\sqrt{3} + 1) \tan \theta + \sqrt{3} = 0$.		
(17) $\cos \theta + \tan \theta = \sec \theta$.	(18) $\sec^2 \theta = 4 \tan \theta$.	

- (19) $3 \operatorname{cosec}^2 \theta = 7 - \tan^2 \theta.$ (20) $\tan^2 \theta - 4 \sec \theta + 5 = 0.$
 (21) $\cos 2\theta + 2 \sin^2 2\theta = 1.$ (22) $\cos 2x = 2 \sin^2 x.$
 (23) $\sin 2x + \sqrt{3} \sin x = 0.$ (24) $\cos 4\theta - \cos 2\theta = -1.$
 (25) $\tan 2\theta \tan 3\theta = 1.$ (26) $\tan 2x + 3 \cot x = 0.$
 (27) $\tan 6\theta + \cot 7\theta = 0.$ (28) $\tan 3\theta = 3 \tan \theta.$
 (29) $\cos 3\theta - \cos 5\theta = \sin \theta.$ (30) $\sin 6\theta + \sin 4\theta = 2 \cos \theta.$
 (31) $\sin 3\theta + \sin 2\theta + \sin \theta = 0.$
 (32) $\cos 5x + \cos 3x + \cos x = 0.$
 (33) $\sin 5x \cos 3x = \sin 9x \cos 7x.$
 (34) $\cot \theta - \operatorname{cosec} 2\theta = 1.$

EXAMPLES 72. MISCELLANEOUS.

(1) Determine the circular measure of an angle of an equilateral triangle.

(2) What is the length of the arc of a circle of 4000 miles radius, which subtends at the centre an angle of 1° ?
 $[\pi = 3.1416.]$

(3) Find how many degrees, minutes, and seconds are passed over by the hour and minute hands of a clock respectively in $2\frac{1}{9}$ minutes.

(4) The angles of a triangle are in A.P., and the number of grades in the least is equal to the number of degrees in the greatest: find the number of degrees in each.

(5) Construct an angle whose cosine is $-\frac{2}{5}$; and find its sine and tangent.

(6) Prove that $\cos 36^\circ = \frac{1}{4}(\sqrt{5} + 1).$

(7) Prove that $\sin 18^\circ \sin 54^\circ = \sin^2 30^\circ.$

(8) Prove that $\sin^2 36^\circ - \sin^2 18^\circ = \sin^2 72^\circ - \sin^2 54^\circ.$

(9) Prove that (i) $\sin 3A \operatorname{cosec} A - \cos 3A \sec A = 2.$

(ii) $\cos A + \cos (120^\circ - A) + \cos (120^\circ + A) = 0.$

(10) Prove that

(i) $4 \sin^6 A + 4 \cos^6 A - 1 = 3 \cos^2 2A.$

(ii) $\frac{\sin A + \sin 3A + \sin 5A + \sin 7A}{\cos A + \cos 3A + \cos 5A + \cos 7A} = \tan 4A.$

(11) Prove that $\log_3 158$ lies between 4 and 5; and that $\log_{158} 3$ lies between .2 and .25.

(12) Given $\log 102 = 2.00860017$ and $\log 102.1 = 2.00902574$, find $\log .0102034$.

(13) Given $\log 125 = 2.0969100$ and $\log 49 = 1.6901961$, find $\log (35)^{\frac{1}{2}}$.

(14) From the definition of a logarithm prove that

$$\log_b a \cdot \log_c b \cdot \log_a c = 1.$$

(15) Shew that in any triangle

$$(i) \quad (b + c) \sin A = 2a \cos \frac{B - C}{2} \cos \frac{A}{2}.$$

$$(ii) \quad (a + c) \tan \frac{B}{2} + (a - c) \cot \frac{B}{2} = 2c \cot C.$$

(16) From a certain point of observation the angle of elevation of the top of a tower is seen to be 45° . The observer, after walking 80 ft. towards the tower, finds the angle of elevation to be 60° . Find the height of the tower to the nearest foot.

(17) Two points P, Q are on the same level with the foot of a tower which stands between them on the straight line PQ. From P and Q the top of the tower is seen at elevations of 45° and 30° respectively. If the distance PQ be 200 yds., find the height of the tower.

(18) A statue on the top of a pillar subtends equal angles at distances 9 yds. and 11 yds. from the foot of the pillar, the tangent of each angle being $\frac{1}{10}$. Find the height of the pillar and of the statue.

(19) The sides of a triangle are in the ratio 4 : 7 : 9. Find the smallest angle to the nearest second. Given $\log 2$,

$$L \tan 12^\circ 37' = 9.3499220, \quad L \tan 12^\circ 36' = 9.3493290.$$

(20) If $a = 5780$, $c = 7639$, $B = 43^\circ 8'$, find A and C. Given

$$\log 185.9 = 2.26928, \quad \log 13.419 = 1.12772,$$

$$L \cot 21^\circ 34' = 10.40312, \quad L \tan 19^\circ 18' 50'' = 9.54468.$$

(21) If $b = 29$, $A = 43^\circ 36' 10''.1$, $C = 124^\circ 58' 33''.6$, find a and B. Given $\log 1.01 = .0043209$,

$$L \sin 43^\circ 36' = 9.8386096, \quad \text{diff. for } 60'' = .0001326,$$

$$L \sin 11^\circ 25' = 9.2965390, \quad \text{diff. for } 60'' = .0006251.$$

- (22) If $a = 13$, $b = 37$, $A = 18^\circ 55' 28'' \cdot 7$, find B . Given

$$\log 37 = 1.5682017, \quad \log 13 = 1.1139434,$$

$$L \sin 18^\circ 55' = 9.5108031, \quad \text{diff. for } 60'' = .0003685,$$

$$L \sin 67^\circ 22' = 9.9651953, \quad \text{diff. for } 60'' = .0000527.$$

- (23) A ship C is observed from two stations A and B on shore. If $AB = 8340$ yards, $\angle BAC = 36^\circ 42'$, $\angle ABC = 118^\circ 22'$, find the distance of A from the ship. Given

$$\log 8.34 = .9211661, \quad \log 1.7408 = .2407489,$$

$$L \sin 61^\circ 38' = 9.9444457, \quad L \sin 24^\circ 56' = 9.6248629.$$

- (24) Prove that the area of a triangle ABC is

$$a^2 \frac{\cos(B-C) + \cos A}{4 \sin A}.$$

- (25) Prove that $r_1 \cos^2 \frac{A}{2} + r_2 \cos^2 \frac{B}{2} + r_3 \cos^2 \frac{C}{2} = \frac{s^2}{2R}$.

- (26) Prove that $\cos A + \cos B \cos C = \frac{bc}{4R^2}$.

- (27) Prove that

$$\frac{r_2 + r_3}{r_1} \tan^2 \frac{A}{2} + \frac{r_3 + r_1}{r_2} \tan^2 \frac{B}{2} + \frac{r_1 + r_2}{r_3} \tan^2 \frac{C}{2} = 2.$$

- (28) ABC is a triangle and D is the middle point of BC : prove that $\cot B \sim \cot C = 2 \cot ADC$.

- (29) Prove that the radius of the incircle of the pedal triangle is $2R \cos A \cos B \cos C$.

- (30) Solve the equations:—

$$(i) \quad \cos 2A = \cos^2 A. \quad (ii) \quad \tan^2 3A = \cot^2 a.$$

$$(iii) \quad \operatorname{cosec} 2x = \cot x. \quad (iv) \quad \cos 12A + 2 = 3 \cos 6A.$$

$$(v) \quad 2 \cos A \cos 3A + 1 = 0. \quad (vi) \quad \sqrt{3} \cos A + \sin A = \sqrt{2}.$$

QUESTIONS ON BOOKWORK.

1. Define (i) an *angle*, (ii) the *measure* of an angle. (Arts. 2, 4, 5.)
2. Describe the two methods which have been adopted for measuring angles. (Arts. 6, 11.)
3. Prove that the circumferences of circles are proportional to their radii. (Art. 12.)
4. Define a *radian*. Prove that a radian is an angle of invariable magnitude. (Arts. 11, 15.)
5. Define the *circular* measure of an angle. Express a radian in degrees, minutes and seconds. (Arts. 11, 16, 22.)
6. If D be the number of degrees, G the number of grades and θ the number of radians in an angle, prove that

$$\frac{D}{180} = \frac{G}{200} = \frac{\theta}{\pi}. \quad (\text{Art. 19.})$$

7. Define the trigonometrical ratios of an angle, and prove that they are always the same for the same angle. What are the *versed* sine and the *covered* sine? (Arts. 25, 26, 28.)
8. Define the *complement* of an angle. State the relations between the trigonometrical ratios of an angle and the trigonometrical ratios of its complement. (Arts. 29, 30.)
9. Determine the values of the trigonometrical ratios of angles of 45° , 60° and 30° . (Art. 31.)
10. Determine the values of the trigonometrical ratios of angles of 0° and 90° . (Art. 34.)
11. What limits are there to the values of the trigonometrical ratios of an angle? For an angle in the first quadrant, which ratios *increase*, and which *decrease*, as the angle increases? (Arts. 36, 37.)
12. With the aid of a figure, shew how any one of the trigonometrical ratios of an angle may be expressed in terms of any other. (Art. 42.)
13. Shew how to determine, geometrically, the other trigonometrical ratios of an angle when any one ratio is given. (Art. 43.)
14. Shew how to construct an acute angle of which any one of the trigonometrical ratios is given. (Art. 44.)
15. Prove geometrically that for any angle A
 - (i) $\tan A = \frac{\sin A}{\cos A}$; (ii) $\sin^2 A + \cos^2 A = 1$;
 - (iii) $1 + \tan^2 A = \sec^2 A$; (iv) $1 + \cot^2 A = \operatorname{cosec}^2 A$. (Art. 46.)
16. What is meant by an *identity*? (Art. 47.)

17. Prove geometrically the formulæ for
 $\sin(A+B)$, $\cos(A+B)$, $\sin(A-B)$, $\cos(A-B)$. (Arts. 49, 52.)
18. From the formulæ for $\sin(A+B)$ and $\cos(A+B)$ deduce a formula for $\tan(A+B)$ in terms of $\tan A$ and $\tan B$. (Art. 50.)
19. From the formulæ for the sine, cosine and tangent of $A+B$, deduce the values of the sine, cosine and tangent of 75° . (Art. 51.)
20. From the formulæ for $\sin(A-B)$ and $\cos(A-B)$ deduce a formula for $\tan(A-B)$ in terms of $\tan A$ and $\tan B$. (Art. 53.)
21. From the formulæ for the sine, cosine and tangent of $A-B$, deduce the values of the sine, cosine and tangent of 15° . (Art. 54.)
22. From the formulæ for $\sin(A \pm B)$ and $\cos(A \pm B)$ deduce expressions for (i) the sum of two sines, (ii) the difference of two sines, (iii) the sum of two cosines, (iv) the difference of two cosines. (Art. 56.)
23. Express each of the following products as the sum or difference of two sines or cosines ;—
 (i) $2 \sin A \cos B$; (ii) $2 \cos A \sin B$;
 (iii) $2 \cos A \cos B$; (iv) $2 \sin A \sin B$. (Art. 57.)
24. Find an expression for $\sin(A+B+C)$ in terms of the sines and cosines of A , B and C . (Art. 58.)
25. Find an expression for $\tan(A+B+C)$ in terms of the tangents of A , B and C . (Art. 58.)
26. Prove (i) $\sin 2A = 2 \sin A \cos A$ (Art. 59);
 (ii) $\cos 2A = \cos^2 A - \sin^2 A$ (Art. 59);
 (iii) $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$ (Art. 59);
 (iv) $\sin 2A = \frac{2 \tan A}{1 + \tan^2 A}$ (Art. 61).
27. Prove (i) $\sin 3A = 3 \sin A - 4 \sin^3 A$;
 (ii) $\cos 3A = 4 \cos^3 A - 3 \cos A$;
 (iii) $\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$. (Art. 62.)
28. Prove that $\sin 18^\circ = \frac{\sqrt{5}-1}{4}$. (Art. 63.)
29. Explain the convention with regard to positive and negative signs when applied to lines and angles. (Arts. 66—70.)
30. State the algebraic signs of the trigonometrical ratios of an angle, taking the quadrants in turn. (Arts. 71, 72.)
31. Determine the trigonometrical ratios of angles of 180° and 270° . (Arts. 74, 75.)
32. Define the *complement* and the *supplement* of an angle. (Arts. 76, 77.)

33. Determine the trigonometrical ratios of the angles (i) $180^\circ - A$, (ii) $180^\circ + A$, (iii) $360^\circ - A$, and (iv) $-A$, in terms of the ratios of the angle A . (Arts. 79—81.)

34. Determine the trigonometrical ratios of the angles (i) $90^\circ - A$, (ii) $90^\circ + A$. (Arts. 85, 86.)

35. Trace the changes in the value of (i) the sine, (ii) the cosine, (iii) the tangent, as the angle increases from 0° to 360° . (Arts. 89—91.)

36. Define a *logarithm*, a *characteristic*, a *mantissa*. (Arts. 93, 97.)

37. Prove that

- (i) $\log_a 1 = 0$, (ii) $\log_a a = 1$, (iii) $\log_a mn = \log_a m + \log_a n$,
 (iv) $\log_a (m \div n) = \log_a m - \log_a n$, (v) $\log_a m^n = n \log_a m$,
 (vi) $\log_b a \times \log_a b = 1$. (Arts. 95, 96.)

38. State how the characteristic of a common logarithm may be determined by inspection. (Arts. 98, 99.)

39. State the advantages of using 10 as the base of a system of logarithms. Prove that any numbers which consist of the same significant figures, in the same order, have the same mantissæ. (Art. 100.)

40. What is the Principle of Proportional Parts? (Art. 108.)

41. Prove the formulæ:—

- (i) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$,
 (ii) $\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1$,
 (iii) $\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C$,
 (iv) $\sin A + \sin B + \sin C = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$,
 (v) $\cos A + \cos B + \cos C = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2} + 1$,

when the sum of A , B and C is 180° . (Art. 118.)

42. Prove the following formulæ connecting the sides and angles of a triangle, and write down the other formulæ analogous to each of them:—

- (i) $a = b \cos C + c \cos B$, (ii) $a^2 = b^2 + c^2 - 2bc \cos A$,
 (iii) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$, (iv) $\cos A = \frac{b^2 + c^2 - a^2}{2bc}$,
 (v) $\sin \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$, (vi) $\cos \frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$,
 (vii) $\tan \frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$, (viii) $\tan \frac{A-B}{2} = \frac{a-b}{a+b} \cot \frac{C}{2}$,
 (ix) $\sin A = \frac{2}{bc} \sqrt{s(s-a)(s-b)(s-c)}$. (Arts. 123—132.)

43. Shew how to solve a triangle, (i) when three sides are given, (ii) when two angles and a side are given, (iii) when two sides and the included angle are given. (Arts. 139—141.)

44. If two sides of a triangle and an angle opposite to one of them be given, determine *geometrically* the conditions that there may be (i) two triangles with the given data, (ii) one triangle only, (iii) no triangle. Shew how to determine the other elements in cases (i) and (ii). (Arts. 142—144.)

45. Determine *algebraically* the conditions required in (44).
(Art. 144.)

46. If a , b , and A be given, and there be two solutions, prove that the sum of the values of c is $2b \cos A$, and that the product is $b^2 - a^2$.
(Art. 144.)

47. Prove the following expressions for the area of a triangle

$$(i) \frac{1}{2}bc \sin A, \quad (ii) \sqrt{s(s-a)(s-b)(s-c)}. \quad (\text{Arts. 145, 146.})$$

48. Prove that in a triangle :—

$$(i) R = \frac{a}{2 \sin A} = \frac{abc}{4\Delta}, \quad (\text{Art. 152})$$

$$(ii) r = \frac{\Delta}{s} = \frac{a}{\cot \frac{B}{2} + \cot \frac{C}{2}}, \quad (\text{Art. 154})$$

$$(iii) r_1 = \frac{\Delta}{s-a} = \frac{a}{\tan \frac{B}{2} + \tan \frac{C}{2}}. \quad (\text{Art. 156.})$$

49. Prove that any chord of a circle is equal to the diameter of the circle multiplied by the sine of the angle in the segment cut off by the chord. (Art. 152, note.)

50. Shew that the sides of the pedal triangle of ABC are $a \cos A$, $b \cos B$, and $c \cos C$, and that the angles opposite to them are $180^\circ - 2A$, $180^\circ - 2B$, and $180^\circ - 2C$, respectively: shew also that the circum-radius of the pedal triangle is $\frac{1}{2}R$. (Art. 160.)

51. Find an expression for all angles which have a given (i) tangent or cotangent, (ii) cosine or secant, (iii) sine or cosecant.
(Arts. 162, 164, 165.)

52. Find an expression for all angles which have two trigonometrical ratios given. (Art. 167.)

53. Shew how to solve an equation of the form $a \sin \theta + b \cos \theta = c$.
(Art. 168.)

ANSWERS.

EXAMPLES 1. PAGE 4.

1. 50 ^{gr} .	2. 16 ^{gr} 66' 66"·6.	3. 13 ^{gr} 33' 33"·3.	4. 25 ^{gr} .	5. 41 ^{gr} .
6. 63 ^{gr} 6' 48"·148.	7. 60 ^{gr} 25'.	8. 60 ^{gr} 0' 74"·074.	9. 3 ^{gr} 12' 50"·.	
10. 41 ^{gr} 11' 48"·148.	11. 23' 33"·3.	12. 24"·691358024.	13. 23° 51'.	
14. 38° 1' 30".	15. 57° 45' 54"·864.	16. 12° 4' 11"·316.		
17. 73° 49' 38"·82.	18. 1° 52' 21"·468.	19. 43° 12' 16"·2.		
20. 1' 22"·296.	21. 1"·62.			

EXAMPLES 2. PAGE 5.

1. 12 $\frac{1}{7}$ in.	2. 31 $\frac{3}{7}$ ft.	3. 22 ft.	4. 39 $\frac{3}{7}$ ft.	5. 29 $\frac{3}{7}$ ft.
6. 22 yds.	7. 1 $\frac{3}{11}$ in.	8. 79 $\frac{6}{11}$ yds.	9. $\frac{7}{176}$ in.	
10. 5 $\frac{1}{3}$ in.	11. 2 $\frac{1}{2}$ in.	12. 1505 yds.		

EXAMPLES 3. PAGE 7.

1. $\frac{\pi}{10}$.	2. $\frac{\pi}{20}$.	3. $\frac{\pi}{6}$.	4. $\frac{\pi}{14}$.	5. $\frac{\pi}{12}$.	6. $\frac{\pi}{18}$.
7. $\frac{\pi}{12}$.	8. $\frac{\pi}{20}$.	9. $\frac{\pi}{6}$.	10. $\frac{\pi}{4}$.	11. $\frac{\pi}{3}$.	12. $\frac{3\pi}{4}$.
13. $\frac{\pi}{10}$.	14. $\frac{\pi}{8}$.	15. $\frac{3\pi}{20}$.	16. $\frac{\pi}{24}$.	17. $\frac{\pi}{8}$.	18. $\frac{\pi}{48}$.
19. $\frac{\pi}{32}$.	20. $\frac{59\pi}{160}$.				

EXAMPLES 4. PAGE 8.

1. 90°.	2. 60°.	3. 45°.	4. 30°.	5. 22° 30'.	6. 36°.
7. 15°.	8. 120°.	9. 135°.	10. 150°.	11. 108°.	
12. 51° 25' 42".	13. 1130° 58' 33".	14. 18°.	15. 286° 28' 12".		
16. 28° 38' 49".	17. 38° 11' 45".	18. 45° 50' 6".	19. 19° 5' 52".		
20. 157° 33' 30".					

EXAMPLES 5. PAGE 9.

1. $\frac{a}{m}$ ft.	2. 2 ft. 4 $\frac{7}{11}$ in.	3. 2 ft. 2 $\frac{4}{11}$ in.	4. 55 ft. 6 $\frac{2}{7}$ in.
5. 2095 $\frac{5}{11}$ miles.	6. 6 ft. 9 $\frac{9}{11}$ in.		

EXAMPLES 6. PAGE 10.

1. 10. 2. $4^{\circ} 30'$. 3. $203\frac{1}{3}^{\circ}$. 4. $7\frac{1}{2}^{\circ}$. 5. $60^{\circ}, 69^{\circ} 54' 44''$.
6. $143^{\circ} 42' 35''$; $54^{\circ} 17' 29''$. 7. $540^{\circ}, 720^{\circ}, 1080^{\circ}, 1440^{\circ}$; $3\pi, 4\pi, 6\pi, 8\pi$.
8. $108^{\circ}, 120^{\circ}, 135^{\circ}, 144^{\circ}$; $\frac{3\pi}{5}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{4\pi}{5}$. 9. $25\frac{5}{7}^{\circ}$; $\frac{\pi}{7}$.
10. $10^{\circ}, 11\frac{1}{3}^{\circ}$. 11. $36^{\circ}, 72^{\circ}, 72^{\circ}$; $\frac{\pi}{5}, \frac{2\pi}{5}, \frac{2\pi}{5}$. 12. $\frac{\pi}{20}, \frac{9\pi}{20}$; $9^{\circ}, 81^{\circ}$.
13. 1.0276 radians. 14. $45^{\circ}, 60^{\circ}, 75^{\circ}$; $\frac{\pi}{4}, \frac{\pi}{3}, \frac{5\pi}{12}$. 15. 24° .
16. 15.278 in. 17. $\sqrt{x^2 - 4}$ in. 18. 7 in. 19. 4 in.
20. $3\sqrt{2}$ in.; 18 sq. in. 21. $\frac{2}{3}\sqrt{2}$ in. 22. $\frac{1}{10}$. 23. 36° , or 40° , or $\frac{\pi}{5}$.
24. $28\frac{3}{2}^{\circ}, 29\frac{3}{2}^{\circ}$. 25. $15^{\circ}, 45^{\circ}$. 26. $\frac{135}{\pi}(\pi - 2)^{\circ}, \frac{3}{4}(\pi - 2)$ radians.
27. $40^{\circ}, 60^{\circ}, 80^{\circ}$. 28. $\frac{1800\pi}{19\pi + 1800}$.

EXAMPLES 7. PAGE 14.

1. 80° . 2. 70° . 3. 37° . 4. 42° . 5. 51° .
6. 18° . 7. $67^{\circ} 30'$. 8. $82^{\circ} 45'$. 9. $46^{\circ} 55'$. 10. $22^{\circ} 10'$.
11. $6^{\circ} 43'$. 12. $64^{\circ} 45' 22''$. 13. $70^{\circ} 31' 7''$. 14. $21^{\circ} 26' 12''$.
15. $84^{\circ} 61'$. 16. $31^{\circ} 46'$. 17. $15^{\circ} 36' 76''$. 18. $62^{\circ} 40' 15''$.
19. $\frac{\pi}{6}$. 20. $\frac{\pi}{3}$. 21. $\frac{3\pi}{8}$. 22. $\frac{\pi}{4}$. 23. $\frac{3\pi}{10}$. 24. $\frac{\pi}{10}$.

EXAMPLES 8. PAGE 15.

1. 55° . 2. 11° . 3. $46^{\circ} 32'$. 4. $2^{\circ} 19'$. 5. $\frac{\pi}{3}$. 6. $\frac{\pi}{6}$.
7. $\frac{\pi}{4}$. 8. $35^{\circ} 12' 28''$. 9. $3^{\circ} 44' 51''$. 10. $31^{\circ} 94' 7''$.

EXAMPLES 9. PAGE 18.

1. 1. 2. $3\frac{1}{3}$. 3. $\frac{4}{3}\sqrt{3}$. 4. 3. 5. 1. 6. $1\frac{1}{3}$. 7. 0.
8. 2. 9. 1. 10. 1. 11. $\frac{1}{4}$. 12. $\frac{1}{3}\sqrt{3}$.

EXAMPLES 10. PAGE 21.

1. $1\frac{1}{4}$. 2. $\frac{1}{4}\sqrt{2}$. 3. 0. 4. $\frac{1}{2}$. 5. 1. 6. $\frac{7}{12}$. 7. 1.
8. 1. 9. 1. 10. $1\frac{3}{4}$. 11. $\frac{1}{4}$. 12. $\frac{3}{8}$.

EXAMPLES 11. PAGE 22.

1. 28 ft. 2. $35\sqrt{3}$ ($=60.62$) ft. 3. $150\sqrt{3}$ ($=259.8$) ft.
4. $\frac{55}{3}\sqrt{3}$ ($=31.75$) ft. 5. 150 ft. 6. $80\sqrt{3}$ ($=138.56$) ft. 7. $\frac{2}{3}$.
8. $215\sqrt{3}$ ($=372.39$) ft.; 430 ft. 9. $40\sqrt{3}$ ($=69.28$) yds.; 80 yds.

10. $\frac{200}{3}\sqrt{3} (=115.47)$ yds.; $\frac{400}{3}\sqrt{3} (=230.94)$ yds. 11. $100\sqrt{3} (=173.2)$ ft.
 12. 450 ft.; $90\sqrt{26} (=458.9)$ ft. 14. $\frac{200}{3}\sqrt{3} (=115.47)$ ft.
 15. $15(3 + \sqrt{3}) \{=70.98\}$ yds.; $30(\sqrt{3} + 1) \{=81.96\}$ yds.
 16. $\frac{17.3}{2}(\sqrt{3} + 1) \{=236.322\}$ ft.; $\frac{17.3}{2}(\sqrt{6} + \sqrt{2}) \{=334.21\}$ ft.
 17. $50(3 - \sqrt{3}) \{=63.4\}$ yds. 18. $880(\sqrt{3} + 1) \{=2404.2\}$ yds.

EXAMPLES 12. PAGE 25.

1. $\sqrt{1 - \cos^2 A}$, $\cos A$, $\frac{\sqrt{1 - \cos^2 A}}{\cos A}$. 2. $\frac{1}{\sqrt{1 + \cot^2 A}}$, $\frac{\cot A}{\sqrt{1 + \cot^2 A}}$, $\frac{1}{\cot A}$.
 3. $\frac{\sqrt{\sec^2 A - 1}}{\sec A}$, $\frac{1}{\sec A}$, $\sqrt{\sec^2 A - 1}$.
 4. $\frac{1}{\operatorname{cosec} A}$, $\frac{\sqrt{\operatorname{cosec}^2 A - 1}}{\operatorname{cosec} A}$, $\frac{1}{\sqrt{\operatorname{cosec}^2 A - 1}}$. 5. $\frac{1}{3}$, $\frac{2}{3}\sqrt{2}$, $\frac{1}{4}\sqrt{2}$.
 6. $\frac{m}{n}$, $\frac{\sqrt{n^2 - m^2}}{n}$, $\frac{m}{\sqrt{n^2 - m^2}}$. 7. $\frac{7}{10}\sqrt{2}$, $\frac{1}{10}\sqrt{2}$, 7.
 8. $\frac{3}{7}\sqrt{73}$, $\frac{3}{7}\sqrt{73}$, $\frac{3}{8}$. 9. $\frac{2}{3}$, $\frac{1}{3}\sqrt{5}$, $\frac{2}{3}\sqrt{5}$.
 10. $\frac{\sqrt{n^2 - m^2}}{n}$, $\frac{m}{n}$, $\frac{\sqrt{n^2 - m^2}}{m}$. 11. $\frac{1}{6}\sqrt{2}$, $\frac{1}{6}\sqrt{23}$, $\frac{1}{23}\sqrt{46}$.
 12. $\frac{1}{6}\sqrt{6}$, $\frac{1}{6}\sqrt{30}$, $\frac{1}{6}\sqrt{5}$. 13. $\frac{1}{3}\sqrt{3}$, $\frac{1}{3}\sqrt{6}$, $\frac{1}{2}\sqrt{2}$.

EXAMPLES 16. PAGE 35.

1. 1, $\frac{1}{2}$, 0, $\frac{\sqrt{3}}{2}$, ∞ , $\frac{1}{\sqrt{3}}$.
 2. $\frac{\sqrt{3} + 1}{2\sqrt{2}}$, $\frac{\sqrt{3} - 1}{2\sqrt{2}}$, $\frac{\sqrt{3} - 1}{2\sqrt{2}}$, $\frac{\sqrt{3} + 1}{2\sqrt{2}}$, $2 + \sqrt{3}$, $2 - \sqrt{3}$.
 3. $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\sqrt{3}$, $\frac{1}{\sqrt{3}}$.
 4. $\frac{\sqrt{3} + 1}{2\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{3} - 1}{2\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, $2 + \sqrt{3}$, 1.
 5. $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{3} - 1}{2\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, $\frac{\sqrt{3} + 1}{2\sqrt{2}}$, 1, $2 - \sqrt{3}$. 6. 1, $\frac{\sqrt{3}}{2}$, 0, $\frac{1}{2}$, ∞ , $\sqrt{3}$.
 7. $\frac{1}{18}(2\sqrt{11} + 5\sqrt{5})$, $\frac{1}{18}(2\sqrt{11} - 5\sqrt{5})$, $\frac{1}{18}(\sqrt{55} - 10)$, $\frac{1}{18}(\sqrt{55} + 10)$,
 $-\frac{1}{6}(8\sqrt{5} + 5\sqrt{11})$, $\frac{1}{6}(5\sqrt{11} - 8\sqrt{5})$.
 8. $\frac{1}{6}\sqrt{65}$, $\frac{7}{6}\sqrt{65}$, $-\frac{1}{6}\sqrt{65}$, $\frac{4}{6}\sqrt{65}$, -8, $1\frac{3}{4}$.
 9. $\frac{1 + \tan A}{1 - \tan A}$. 10. $\frac{1 - \tan A}{1 + \tan A}$. 11. $\frac{\sqrt{3} - \tan A}{1 + \sqrt{3} \tan A}$.
 12. $\frac{\sqrt{3} \tan A - 1}{\sqrt{3} + \tan A}$. 13. $\frac{\tan A - (2 - \sqrt{3})}{1 + (2 - \sqrt{3}) \tan A}$. 14. $\frac{1 - (2 + \sqrt{3}) \tan A}{\tan A + 2 + \sqrt{3}}$.

EXAMPLES 17. PAGE 37.

1. $2 \sin 40^\circ \cos 10^\circ$.	2. $2 \sin 40^\circ \cos 20^\circ$.	3. $2 \sin 35^\circ \cos 5^\circ$.
4. $2 \sin 45^\circ \cos 25^\circ$.	5. $2 \sin 45^\circ \cos 15^\circ$.	6. $2 \cos 35^\circ \sin 5^\circ$.
7. $2 \cos 37^\circ \sin 9^\circ$.	8. $2 \cos 35^\circ \sin 24^\circ$.	9. $2 \cos 29^\circ \sin 6^\circ$.
10. $2 \cos 30^\circ \cos 27^\circ$.	11. $2 \cos 37^\circ \cos 9^\circ$.	12. $2 \cos 25^\circ \cos 5^\circ$.
13. $2 \cos 43^\circ \cos 33^\circ$.	14. $2 \cos 38^\circ \cos 4^\circ$.	15. $2 \sin 75^\circ \sin 5^\circ$.
16. $2 \sin 31^\circ \sin 8^\circ$.	17. $2 \sin 37^\circ \sin 1^\circ$.	18. $2 \sin 45^\circ \sin 14^\circ$.
19. $2 \sin 29^\circ \sin 6^\circ$.	20. $2 \sin 18^\circ \sin 15^\circ$.	21. $2 \sin 38^\circ \sin 10^\circ$.
22. $-2 \sin 37\frac{1}{2}^\circ \sin 17\frac{1}{2}^\circ$.	23. $-2 \sin 42\frac{1}{2}^\circ \sin 3\frac{1}{2}^\circ$.	24. $2 \sin \frac{7\pi}{20} \cos \frac{3\pi}{20}$.
25. $2 \cos \frac{5\pi}{21} \sin \frac{2\pi}{21}$.	26. $2 \cos \frac{7\pi}{24} \cos \frac{\pi}{24}$.	27. $2 \cos \frac{13\pi}{48} \sin \frac{5\pi}{48}$.
28. $2 \sin 2A \cos A$.	29. $2 \cos 2A \cos A$.	30. $2 \sin 2A \sin A$.
31. $2 \sin \frac{5A}{2} \sin \frac{A}{2}$.	32. $2 \sin \frac{9A}{2} \cos \frac{A}{2}$.	33. $2 \cos \frac{9A}{2} \sin \frac{A}{2}$.
34. $2 \sin \frac{15A}{2} \sin \frac{3A}{2}$.	35. $2 \cos A \sin B$.	36. $2 \cos A \cos B$.
37. $2 \sin A \cos B$.	38. $2 \sin A \sin B$.	39. $2 \cos nA \cos A$.
40. $2 \cos \frac{2n+1}{2} A \sin \frac{A}{2}$.		

EXAMPLES 18. PAGE 37.

5. $\frac{\sin 90^\circ}{\cos 60^\circ \cos 30^\circ}$.	6. $\frac{\sin 45^\circ}{\cos 60^\circ \cos 15^\circ}$.	7. $\frac{\sin 60^\circ}{\cos 75^\circ \cos 15^\circ}$.
8. $\frac{\sin 65^\circ}{\cos 25^\circ \cos 40^\circ}$.	9. $\frac{\sin 60^\circ}{\sin 15^\circ \sin 75^\circ}$.	10. $\frac{\sin 15^\circ}{\sin 60^\circ \sin 75^\circ}$.
11. $\frac{\cos 16^\circ}{\sin 42^\circ \cos 26^\circ}$.	12. $\frac{\cos 67^\circ}{\sin 28^\circ \cos 39^\circ}$.	13. $\frac{\sin 8A}{\cos 2A \cos 6A}$.
14. $\frac{\sin 2A}{\cos 5A \cos 3A}$.	15. $\frac{\sin A}{\sin 2A \sin 3A}$.	16. $\frac{\cos 2A}{\cos A \sin 3A}$.

EXAMPLES 19. PAGE 39.

1. $\sin 67^\circ + \sin 33^\circ$.	2. $\sin 69^\circ + \sin 53^\circ$.	3. $\sin 61^\circ + \sin 5^\circ$.
4. $\sin 89^\circ + \sin 69^\circ$.	5. $\sin 33^\circ + \sin 21^\circ$.	6. $\sin 85^\circ - \sin 25^\circ$.
7. $\sin 85^\circ - \sin 57^\circ$.	8. $\sin 39^\circ - \sin 1^\circ$.	9. $\sin 89^\circ - \sin 83^\circ$.
10. $\sin 53^\circ - \sin 15^\circ$.	11. $\cos 85^\circ + \cos 13^\circ$.	12. $\cos 71^\circ + \cos 47^\circ$.
13. $\cos 64^\circ + \cos 28^\circ$.	14. $\cos 49^\circ + \cos 1^\circ$.	15. $\cos 14^\circ + \cos 2^\circ$.
16. $\cos 3^\circ - \cos 83^\circ$.	17. $\cos 4^\circ - \cos 8^\circ$.	18. $\cos 5^\circ - \cos 53^\circ$.
19. $\cos 75^\circ - \cos 81^\circ$.	20. $\cos 27^\circ - \cos 37^\circ$.	21. $\sin 11A + \sin 5A$.
22. $\sin 5A + \sin 3A$.	23. $\sin 16A + \sin 4A$.	24. $\sin 15A + \sin A$.
25. $\sin 7A + \sin 3A$.	26. $\sin 7A - \sin 5A$.	27. $\sin 11A - \sin 5A$.
28. $\sin 9A - \sin A$.	29. $\sin 3A - \sin A$.	30. $\sin 12A - \sin 2A$.
31. $\cos 10A + \cos 6A$.	32. $\cos 11A + \cos 3A$.	33. $\cos 11A + \cos A$.

34. $\cos A - \cos 5A$. 35. $\cos A - \cos 17A$.
 36. $\cos(A+B) + \cos(A-B)$. 37. $\cos(A-B) - \cos(A+B)$.
 38. $\sin(A+B) - \sin(A-B)$. 39. $\cos(m+n)A + \cos(m-n)A$.
 40. $\cos(m-n)A - \cos(m+n)A$.

EXAMPLES 20. PAGE 40.

1. $\sin A \sin B \sin C + \sin A \cos B \cos C + \cos A \sin B \cos C - \cos A \cos B \sin C$.
 2. $\cos A \cos B \cos C - \cos A \sin B \sin C - \sin A \cos B \sin C - \sin A \sin B \cos C$.
 3. $\cos A \cos B \cos C + \cos A \sin B \sin C - \sin A \cos B \sin C + \sin A \sin B \cos C$.
 4. $\sin A \cos B \cos C \cos D + \cos A \sin B \cos C \cos D + \cos A \cos B \sin C \cos D$
 $+ \cos A \cos B \cos C \sin D - \cos A \sin B \sin C \sin D - \sin A \cos B \sin C \sin D$
 $- \sin A \sin B \cos C \sin D - \sin A \sin B \sin C \cos D$.
 5. $1\frac{2}{3}$. 6. $2\frac{5}{11}$. 7. 13. 8. $1\frac{1}{4}$.

EXAMPLES 24. PAGE 47.

1. $\frac{7}{8}\pi$. 2. 7 ft. 3. 5 in. 4. $24^\circ 32' 43\frac{7}{11}''$. 5. $10; 40^\circ, 50^\circ, 90^\circ$.
 6. $\frac{4}{5}, \frac{3}{5}, \frac{4}{5}$. 7. $\frac{1}{5}$; 4. 8. $53^\circ 18' 3''; \frac{\pi}{14}$. 9. 15 in.
 25. $40\sqrt{3} (=69.28)$ ft.

EXAMPLES 25. PAGE 55.

The ratios which are negative are:—in 1st quadrant, none; in 2nd quadrant, cosine, tangent, secant and cotangent; in 3rd quadrant, sine, cosine, cosecant and secant; in 4th quadrant, sine, tangent, cotangent, cosecant.

We state below the number of the quadrant in which the angles lie.

1. 1. 2. 2. 3. 1. 4. 2. 5. 2. 6. 3. 7. 1. 8. 4.
 9. 4. 10. 2. 11. 4. 12. 3. 13. 4. 14. 3. 15. 2. 16. 4.
 17. 4. 18. 2. 19. 1. 20. 4. 21. 3. 22. 1. 23. 4. 24. 3.
 25. 2. 26. 3. 27. 3. 28. 2. 29. 1. 30. 4. 31. 2. 32. 2.
 33. 3. 34. 3.

EXAMPLES 26. PAGE 56.

1. $67^\circ, 157^\circ$. 2. $70^\circ 51' 39'', 160^\circ 51' 39''$. 3. $139^\circ, 229^\circ$.
 4. $\frac{3\pi}{8}, \frac{7\pi}{8}$. 5. $\frac{\pi}{14}, \frac{4\pi}{7}$. 6. $\frac{5\pi}{8}, \frac{9\pi}{8}$. 7. $127^\circ 58', 217^\circ 58'$.
 8. $173^\circ 14' 55'', 263^\circ 14' 55''$. 9. $131^\circ 5' 7'', 231^\circ 5' 7''$.
 10. $-128^\circ, -38^\circ$. 11. $-105^\circ 42', -15^\circ 42'$.
 12. $-135^\circ 33' 54'', -45^\circ 33' 54''$. 13. $-\frac{7\pi}{6}, -\frac{2\pi}{3}$. 14. $-\frac{13\pi}{16}, -\frac{5\pi}{16}$.
 15. $\frac{5\pi}{3}, \frac{13\pi}{6}$. 16. $510^\circ, 600^\circ$. 17. $-\left(\frac{\pi}{2} + A\right), -A$.
 18. $-\frac{\pi}{6} + A, \frac{\pi}{3} + A$.

EXAMPLES 27. PAGE 59.

1. $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1.$ 2. $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1.$ 3. $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1.$
4. $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}.$ 5. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}.$ 6. $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}.$
7. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}.$ 8. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}.$ 9. $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}.$
10. $\frac{\sqrt{3}+1}{2\sqrt{2}}, -\frac{\sqrt{3}-1}{2\sqrt{2}}, -(2+\sqrt{3}).$ 11. $-\frac{\sqrt{3}+1}{2\sqrt{2}}, -\frac{\sqrt{3}-1}{2\sqrt{2}}, 2+\sqrt{3}.$
12. $-\frac{\sqrt{3}+1}{2\sqrt{2}}, \frac{\sqrt{3}-1}{2\sqrt{2}}, -(2+\sqrt{3}).$ 13. $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1.$
14. $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}.$ 15. $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}.$
16. $-\frac{\sqrt{3}-1}{2\sqrt{2}}, \frac{\sqrt{3}+1}{2\sqrt{2}}, -(2-\sqrt{3}).$ 17. $-\frac{\sqrt{3}+1}{2\sqrt{2}}, \frac{\sqrt{3}-1}{2\sqrt{2}}, -(2+\sqrt{3}).$
18. $-1, 0, -\infty.$ 19. $\frac{\sqrt{5}-1}{4}.$ 20. $-\frac{\sqrt{5}-1}{4}.$ 21. $-\frac{\sqrt{5}-1}{4}.$

EXAMPLES 28. PAGE 60.

1. $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}.$ 2. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}.$ 3. $-\frac{\sqrt{3}}{2}, -\frac{1}{2}, \sqrt{3}.$
4. $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1.$ 5. $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1.$ 6. $-\frac{1}{2}, \frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}.$
7. $\frac{\sqrt{3}+1}{2\sqrt{2}}, -\frac{\sqrt{3}-1}{2\sqrt{2}}, -(2+\sqrt{3}).$ 8. $\frac{\sqrt{3}-1}{2\sqrt{2}}, -\frac{\sqrt{3}+1}{2\sqrt{2}}, -(2-\sqrt{3}).$
9. $-\frac{\sqrt{3}-1}{2\sqrt{2}}, -\frac{\sqrt{3}+1}{2\sqrt{2}}, 2-\sqrt{3}.$ 10. $-\frac{\sqrt{3}+1}{2\sqrt{2}}, -\frac{\sqrt{3}-1}{2\sqrt{2}}, 2+\sqrt{3}.$
11. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}.$ 12. $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1.$ 13. $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}.$
14. $-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 1.$ 15. $-\frac{\sqrt{3}-1}{2\sqrt{2}}, -\frac{\sqrt{3}+1}{2\sqrt{2}}, 2-\sqrt{3}.$
16. $\frac{\sqrt{3}}{2}, -\frac{1}{2}, -\sqrt{3}.$ 17. $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -1.$ 18. $-\frac{\sqrt{3}}{2}, \frac{1}{2}, -\sqrt{3}.$
19. $-\frac{1}{2}, -\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}.$ 20. $-1, 0, -\infty.$ 21. $\frac{1}{\sqrt{2}}.$ 22. $\frac{\sqrt{3}}{2}.$

23. $\sqrt{3}$. 24. $\frac{\sqrt{5}-1}{4}$. 25. $2-\sqrt{3}$. 26. $\frac{\sqrt{3}}{2}$. 27. $\frac{\sqrt{3}}{2}$.
 28. 1. 29. $-\frac{1}{2}$. 30. $-\frac{1}{2}$. 31. $\frac{2}{\sqrt{3}}$. 32. $-\frac{2}{\sqrt{3}}$. 33. 1.
 34. $-\frac{\sqrt{3}}{2}$. 35. $-\frac{1}{\sqrt{2}}$.

EXAMPLES 29. PAGE 61.

1. $\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$, $-\frac{1}{\sqrt{3}}$. 2. $-\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, -1 . 3. $-\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $-\sqrt{3}$.
 4. $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $-\frac{\sqrt{3}+1}{2\sqrt{2}}$, $-(2-\sqrt{3})$. 5. $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $2+\sqrt{3}$.
 6. $-\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, 1. 7. $-\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $\sqrt{3}$. 8. $-\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$.
 9. $-\frac{\sqrt{3}-1}{2\sqrt{2}}$, $-\frac{\sqrt{3}+1}{2\sqrt{2}}$, $2-\sqrt{3}$. 10. $-\frac{\sqrt{3}+1}{2\sqrt{2}}$, $-\frac{\sqrt{3}-1}{2\sqrt{2}}$, $2+\sqrt{3}$.
 11. $\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $-\sqrt{3}$. 12. $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, -1 . 13. $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$.
 14. $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, 1. 15. $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$. 16. $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $2-\sqrt{3}$.
 17. $-\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, -1 . 18. $-\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $-\sqrt{3}$. 19. $-\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$.
 20. $\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$, $-\frac{1}{\sqrt{3}}$. 21. $\frac{1}{\sqrt{2}}$. 22. $-\frac{1}{2}$. 23. $-\sqrt{3}$. 24. $-\frac{\sqrt{5}-1}{4}$.
 25. $-(2-\sqrt{3})$. 26. 2. 27. 2. 28. $-\frac{\sqrt{3}}{2}$. 29. $-\frac{1}{2}$.
 30. $\frac{1}{\sqrt{3}}$. 31. -1 . 32. $-\frac{1}{2}$. 33. 2. 34. $\sqrt{2}$. 35. $\sqrt{3}$.

EXAMPLES 30. PAGE 64.

1. $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, 1. 2. $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, -1 . 3. $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, $\sqrt{3}$.
 4. $\frac{\sqrt{3}}{2}$, $-\frac{1}{2}$, $-\sqrt{3}$. 5. $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, $\frac{1}{\sqrt{3}}$. 6. $\frac{1}{2}$, $-\frac{\sqrt{3}}{2}$, $-\frac{1}{\sqrt{3}}$.
 7. $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $-\frac{\sqrt{3}-1}{2\sqrt{2}}$, $-(2+\sqrt{3})$. 8. $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $\frac{\sqrt{3}+1}{2\sqrt{2}}$, $2-\sqrt{3}$.
 9. $\frac{\sqrt{3}-1}{2\sqrt{2}}$, $-\frac{\sqrt{3}+1}{2\sqrt{2}}$, $-(2-\sqrt{3})$. 10. $\frac{1}{\sqrt{2}}$, $\frac{1}{\sqrt{2}}$, 1.

11. $\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -1.$ 12. $\frac{1}{2}, -\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{3}}.$ 13. $\frac{\sqrt{5}-1}{4}.$
 14. $-\frac{\sqrt{5}-1}{4}.$ 15. $-\frac{\sqrt{5}-1}{4}.$ 16. $-\sin A.$ 17. $\sin A.$
 18. $-\cos A.$ 19. $-\cos A.$ 20. $\cos A.$ 21. $\sin A.$ 22. $\cos A.$
 23. $-\sin A.$ 24. $\sec A.$ 25. $\operatorname{cosec} A.$ 26. $-\operatorname{cosec} A.$
 27. $\tan A.$ 28. $-\tan A.$ 29. $-\cot A.$ 30. $\cot A.$

EXAMPLES 31. PAGE 68.

1. 3. 2. 2. 3. $\frac{1}{6}.$ 4. $1\frac{1}{2}.$ 5. $2\frac{1}{2}.$ 6. $\frac{3}{4}.$ 7. $1\frac{1}{3}.$
 8. $\frac{1}{2}.$ 9. $\frac{3}{8}.$ 10. $-8.$ 11. $-\frac{1}{2}.$ 12. $-\frac{2}{3}.$ 13. $-1.$
 14. 12. 15. $\pm 6.$ 16. $\pm 3.$ 17. 9. 18. 512.
 19. $\pm 27.$ 20. $\frac{1}{8\frac{1}{2}}.$

EXAMPLES 32. PAGE 70.

12. $\log a + \log b + 2 \log c - 3 \log d.$ 13. $\log 2 + \log \sin A + \log \cos A.$
 14. $\frac{2}{3}(\log a + 2 \log b - 4 \log c).$ 15. $\frac{2}{3}(\log 2 + \log 3).$
 16. $\log 3 + \log 7 - \log 5.$ 17. $\frac{1}{3}(5 \log 3 - \log 2 - \log 127).$

EXAMPLES 33. PAGE 73.

1. $\overline{13}\cdot4870586.$ 2. $\overline{4}\cdot2687956.$ 3. $1\cdot3521323.$ 4. $\overline{3}\cdot51351.$
 5. $10\cdot78142.$ 6. $2\cdot03622.$ 7. $\overline{9}\cdot6252, \overline{14}\cdot0420, \overline{28}\cdot0840.$
 8. $\overline{4}\cdot4636, \overline{5}\cdot8742, \overline{38}\cdot2784.$ 9. $\overline{21}\cdot9388, \overline{46}\cdot8623, \overline{241}\cdot2656.$
 10. $\overline{1}\cdot2316, \overline{1}\cdot4023, \overline{1}\cdot5110.$ 11. $\overline{2}\cdot10852, \overline{2}\cdot8178, \overline{1}\cdot2434.$

EXAMPLES 34. PAGE 74.

1. $\cdot69897.$ 2. $\cdot9542426.$ 3. $2\cdot4771213.$ 4. $1\cdot39794.$
 5. $2\cdot3344539.$ 6. $1\cdot8573326.$ 7. $2\cdot1583626.$ 8. $2\cdot0334239.$
 9. $3\cdot0334239.$ 10. $-2.$ 11. $\overline{4}\cdot30103.$ 12. $\overline{3}\cdot6812413.$
 13. $\overline{4}\cdot9822713.$ 14. $\cdot6532126.$ 15. $\cdot3802113.$ 16. $\overline{6}\cdot2552726.$
 17. $\overline{8}\cdot77528.$ 18. $\overline{1}\cdot2730013.$ 19. $\overline{1}\cdot849485.$ 20. $\cdot0624694.$
 21. $\overline{1}\cdot7406163.$ 22. $\cdot1023132.$ 23. $\overline{1}\cdot0263938.$ 24. $\overline{1}\cdot7108823.$

EXAMPLES 35. PAGE 74.

1. 2. 2. 3. 3. 4. 4. 5. 5. 7. 6. $-1.$ 7. $-2.$
 8. $-3.$ 9. $-4.$ 10. $-9.$ 11. $-14.$ 12. $-2.$ 13. $-3.$
 14. $-2.$ 15. $-4.$ 16. 208. 17. 353. 18. 125. 19. 69.
 20. 3560. 21. 60. 22. 38. 23. 59. 24. 17. 25. 4225.

EXAMPLES 36. PAGE 75.

1. 1. 2. (i) $\frac{1}{2}$; (ii) $-\frac{1}{2}$; (iii) 4. 3. 2, -6 , 4.
 4. (i) 1000 & 10000; (ii) 1 & 10; (iii) $\cdot001$ & $\cdot0001.$
 5. 4, 512, 4096, 256, 128. 6. $\overline{5}\cdot0211893.$

7. (i) $1\cdot6232493$; (ii) $\bar{1}\cdot9242793$; (iii) $\bar{3}\cdot5228787$; (iv) $\cdot5880456$;
 (v) $\bar{1}\cdot7489455$; (vi) $\bar{1}\cdot5369447$. 8. 16, 22, 118, 4. 9. 9.
 10. Greater. 11. $3 - 6 \log 5$. 12. (i) $\cdot30103$; (ii) $\cdot2916$.
 13. $\cdot6989700$, $\cdot30103$, $\bar{1}\cdot9880618$. 14. $3\cdot30193$.
 15. $1\cdot732393$, $\bar{1}\cdot7781513$. 16. 1.

EXAMPLES 37. PAGE 78.

1. $\cdot8330929$. 2. $\cdot3223192$. 3. $\bar{4}\cdot6364625$. 4. $7\cdot9360196$.

EXAMPLES 38. PAGE 78.

1. $4\cdot86507$. 2. $8\cdot479382$. 3. $70\cdot30548$. 4. $\cdot02411077$.

EXAMPLES 39. PAGE 79.

1. $\cdot3948329$. 2. $\cdot7436377$. 3. $\cdot6320381$. 4. $\cdot5472667$.
 5. $1\cdot3522366$.

EXAMPLES 40. PAGE 80.

1. $54^\circ 36' 38''\cdot03$. 2. $16^\circ 17' 21''\cdot07$. 3. $62^\circ 23' 46''\cdot24$.
 4. $58^\circ 2' 25''\cdot32$. 5. $26^\circ 45' 32''\cdot54$.

EXAMPLES 41. PAGE 81.

1. $\cdot9863695$; $9^\circ 28' 37''\cdot2$. 2. $\cdot8542245$; $31^\circ 19' 20''\cdot38$.
 3. $\cdot1444436$; $81^\circ 46' 33''\cdot9$. 4. $\cdot5005020$; $63^\circ 24' 38''\cdot9$.
 5. $1\cdot1848865$; $57^\circ 33' 30''\cdot1$.

EXAMPLES 42. PAGE 82.

1. $9\cdot6137596$; $24^\circ 15' 33''\cdot9$. 2. $10\cdot3827281$; $67^\circ 29' 27''\cdot9$.
 3. $10\cdot2393442$; $54^\circ 48' 19''\cdot7$. 4. $9\cdot4217565$; $74^\circ 41' 15''$.
 5. $9\cdot6985583$; $63^\circ 27' 39''\cdot6$. 6. $10\cdot2168925$; $37^\circ 21' 33''\cdot9$.

EXAMPLES 43. PAGE 83.

1. $\cdot5105452$. 2. $\bar{1}\cdot39794$; $9\cdot69897$; $10\cdot1505150$.
 3. $2\cdot30103$; $\cdot59862$; $9\cdot69897$; $9\cdot849485$. 4. $1\cdot8$.
 5. $5\cdot6954904$; $\bar{4}\cdot6954817$; $4\cdot6954839$.
 6. $\bar{3}\cdot5651392$; $4\cdot5651404$; $3\cdot5651451$. 7. 35. 8. $\bar{1}\cdot7962206$.
 9. $1\cdot3081$. 10. $\cdot63865$. 11. $\cdot22221$. 12. $\cdot60020$.
 13. $1\cdot02985$. 14. 5, 2. 15. $9\cdot9849438$.

EXAMPLES 46. PAGE 89.

1. $\sin 2A + \sin 2B + \sin 2C$. 2. $1 - \cos 2A - \cos 2B + \cos 2C$.
 3. $-\sin 2A + \sin 2B + \sin 2C$. 4. $-1 - \cos 2A - \cos 2B - \cos 2C$.
 5. $\sin 2A - \sin 2B + \sin 2C$. 6. $\cos A + \cos B + \cos C - 1$.
 7. $\sin A + \sin B - \sin C$. 8. $1 - \cos A + \cos B + \cos C$.
 9. $\sin A + \sin B + \sin C$. 10. $1 + \cos A + \cos B - \cos C$.

EXAMPLES 48. PAGE 92.

1. $\sqrt{3} + 1$. 2. 20. 3. 6. 4. $2\sqrt{6}$. 5. $4(\sqrt{3} + 1)$. 6. $3(\sqrt{6} - \sqrt{2})$.

EXAMPLES 49. PAGE 94.

1. $\frac{5}{3}\sqrt{6}$. 2. $22\sqrt{2}$. 3. $\frac{19}{2}(\sqrt{6} - \sqrt{2})$. 4. $\frac{25}{2}\sqrt{2}$.
5. $\frac{13}{2}(3\sqrt{2} - \sqrt{6})$. 6. $\frac{3}{2}(\sqrt{6} - \sqrt{2})$.

EXAMPLES 50. PAGE 95.

1. 1. 2. 1. 3. 3.46. 4. 4.12. 5. 9.539.
6. 7.21. 7. 1.88.

EXAMPLES 51. PAGE 96.

1. $\frac{1}{2}$. 2. $\frac{1}{\sqrt{2}}$. 3. 0. 4. $\frac{1}{2}\sqrt{3}$. 5. $\frac{1}{3}\frac{3}{8}$. 6. $\frac{1}{3}\frac{3}{8}$. 7. $\frac{1}{5}$.
8. $\frac{7}{8}$. 9. $-\frac{1}{4}$. 10. $\frac{3}{2}\frac{3}{2}\frac{3}{2}$.

EXAMPLES 52. PAGE 98.

1. $\frac{1}{3}, \frac{2}{3}\sqrt{2}, \frac{1}{4}\sqrt{2}, \frac{1}{4}\sqrt{2}$. 2. $\frac{1}{4}\sqrt{7}, \frac{3}{4}, \frac{1}{3}\sqrt{7}, \frac{3}{8}\sqrt{7}$. 3. $\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 1, 1$.
4. $\frac{1}{2}\sqrt{2}, \frac{1}{2}\sqrt{2}, 1, 1$. 5. $\frac{1}{2}\sqrt{3}, \frac{1}{2}, \sqrt{3}, \frac{1}{2}\sqrt{3}$. 6. $\frac{1}{2}, \frac{1}{2}\sqrt{3}, \frac{1}{3}\sqrt{3}, \frac{1}{2}\sqrt{3}$.

EXAMPLES 54. PAGE 101.

3. $A=B$. 10. $\frac{3}{5}$.

EXAMPLES 55. PAGE 104.

1. $B=15^\circ; a=38.6370; b=10.3528$. 2. $B=74^\circ 13'; a=5.9840; b=21.1705$.
3. $B=42^\circ 55'; a=15.0570; c=20.56$. 4. $B=30^\circ; a=5.6292; c=6.5$.
5. $B=53^\circ 50'; b=34.1999; c=42.3631$.
6. $B=11^\circ 26'; b=3.0336; c=15.3037$.

EXAMPLES 56. PAGE 105.

1. $A=36^\circ 52' 11''.6; B=53^\circ 7' 48''.4; c=5$.
2. $A=64^\circ 13' 42''.7; B=25^\circ 46' 17''.3; a=20.71$.
3. $A=33^\circ 41' 24''.2; B=56^\circ 18' 35''.8; c=9.0139$.
4. $A=75^\circ 57' 49''.5; B=14^\circ 2' 10''.5; c=38.1387$.

EXAMPLES 57. PAGE 107.

1. $1\frac{7}{8}$. 2. $-\frac{1}{16}$. 5. $82^\circ 49'$. 6. 45° . 7. $44^\circ 24' 55''$.
8. $\frac{3}{8}$. 9. $104^\circ 28' 39''$. 10. $43^\circ 2' 56''.68$. 11. $48^\circ 11' 22''.86$.
12. $55^\circ 46' 16''.1$. 13. $20^\circ 2' 59''.5$.

EXAMPLES 58. PAGE 108.

1. $a=26.8701; b=9.8351$. 2. 172.6436 ft. 3. 25.98716 ft.
4. 14.3595 ft., 19.3465 ft.

EXAMPLES 59. PAGE 110.

1. 9.8489. 2. $3; \frac{7}{8}; \frac{1}{8}$. 3. $7; \frac{13}{14}; \frac{11}{14}$. 4. 60° . 5. $\frac{1}{3}\sqrt{7}$.
6. $105^\circ, 15^\circ$. 7. $\sqrt{13}; \frac{1}{26}\sqrt{39}; \frac{1}{4}\sqrt{3}$. 8. $\frac{7}{2}\sqrt{3}, 30^\circ, 90^\circ$.
9. $10\sqrt{3}, 90^\circ, 30^\circ$. 10. $65^\circ 38' 12''; 61^\circ 59' 48''$.
11. $116^\circ 33' 54''; 26^\circ 33' 54''$. 12. $117^\circ 38' 45''; 27^\circ 38' 45''$.
13. $60^\circ 6' 2''\cdot 4; 49^\circ 53' 57''\cdot 6$. 14. $101^\circ 34' 1''; 18^\circ 25' 59''$.
15. $49^\circ 19' 30''\cdot 4; 40^\circ 40' 29''\cdot 6$. 16. $93^\circ 16' 26''\cdot 5; 49^\circ 10' 13''\cdot 5$.

EXAMPLES 60. PAGE 114.

1. One. 2. Two. 3. None. 4. One.
5. $B=60^\circ, C=90^\circ, b=\frac{3\sqrt{3}}{4}\sqrt{3}$. 6. $B=30^\circ, C=30^\circ, c=1$.
7. $B=15^\circ, C=60^\circ, b=\sqrt{3}-1$.
8. $C=60^\circ, B=90^\circ, b=2; C=120^\circ, B=30^\circ, b=1$.
9. $33^\circ 2' 25''\cdot 5$. 10. $27^\circ 2' 35''\cdot 8$. 11. $49^\circ 16' 5''\cdot 5; 10^\circ 43' 54''\cdot 5$.

EXAMPLES 61. PAGE 116.

1. 12. 2. 21. 3. $8\frac{3}{4}$. 4. 18. 5. $35\sqrt{3}$. 6. $\frac{9\sqrt{2}}{4}\sqrt{2}$.
7. 54. 8. 84. 9. $18\sqrt{3}$. 10. $64(3-\sqrt{3})$. 11. 37.0999 yds.

EXAMPLES 62. PAGE 119.

1. 5729 ft. 2. 24.4 yds. 3. 30 ft., 30 ft., 51.96 ft.
4. 57.73 yds.; 115.46 yds. 5. 10.1033 yds. 6. 259 ft. 10 in.
7. 680.34 ft. 8. 76.9 ft. 9. 73.08 ft.; 95.16 ft.
10. $2\frac{1}{2}$ miles or 4400 yds. 11. 34.236 ft. 12. 1558.8 ft.
13. 7212.48 ft. 14. 1524.16 yds.
15. $\sqrt{3}$ miles [=1 mile 1288 yds.]; $1\frac{1}{2}$ mile.
16. 155 ft. 6 in. 17. 237 ft.; 137 ft.
18. $\sqrt{c^2-a^2}$ ft.; $(a+\sqrt{c^2-a^2})$ ft. 19. 45 ft. 20. $\frac{1}{10\sqrt{2}}$. 21. 50 ft.
22. 30 ft. 23. 107.2 ft. 24. 261.2 ft. 25. 12.68 ft.
26. 30 yds. 27. 146.67 ft. 28. 68.2 ft. 29. $(h \tan \alpha \cot \beta)$ ft.
30. $\frac{c \cos (\alpha + \beta) \sin \alpha}{\sin \beta}$.

EXAMPLES 63. PAGE 126.

1. $2\frac{1}{2}$ ft. 2. 10 in. 3. $3\sqrt{2}$ ft. 4. $6\sqrt{3}$ ft. 5. 20 ft.
6. $17(\sqrt{6}+\sqrt{2})$ ft. 7. 6, $2\frac{1}{2}$, 1, 2, 3, 6. 8. 48, $6\frac{1}{2}$, 3, 8, 12, 8.
9. $2\sqrt{5}, \frac{9}{10}\sqrt{5}, \frac{2}{3}\sqrt{5}, \sqrt{5}, \sqrt{5}, 2\sqrt{5}$.
10. $\frac{7\sqrt{15}}{4}, \frac{8}{3}\sqrt{15}, \frac{5}{6}\sqrt{15}, \frac{3}{2}\sqrt{15}, \frac{5}{2}\sqrt{15}, \frac{1}{2}\sqrt{15}$.
11. $8(2\sqrt{3}-3), 8(2\sqrt{3}+3), 8, 8$.
12. $5(\sqrt{3}-1), 5(3+\sqrt{3}), 5(3-\sqrt{3}), 5(\sqrt{3}+1)$.
13. $\frac{1}{2}(2-\sqrt{2}), \frac{1}{2}\sqrt{2}, \frac{1}{2}(2+\sqrt{2}), \frac{1}{2}\sqrt{2}$. 14. $7\frac{1}{2}, 22\frac{1}{2}, 22\frac{1}{2}, 22\frac{1}{2}$.

EXAMPLES 66. PAGE 134.

1. $n\pi + \frac{\pi}{3}$.
2. $n\pi + \frac{2\pi}{3}$.
3. $n\pi + \frac{\pi}{4}$.
4. $n\pi + \frac{3\pi}{4}$.
5. $n\pi$.
6. $n\pi + \frac{\pi}{6}$.
7. $n\pi + \frac{5\pi}{6}$.
8. $n\pi + \frac{3\pi}{4}$.
9. $n\pi + \frac{\pi}{4}$ and $n\pi + \frac{3\pi}{4}$.
10. $n\pi + \frac{\pi}{3}$, $n\pi + \frac{2\pi}{3}$.

EXAMPLES 67. PAGE 135.

1. $2n\pi \pm \frac{\pi}{3}$.
2. $2n\pi \pm \frac{2\pi}{3}$.
3. $2n\pi \pm \frac{\pi}{6}$.
4. $2n\pi \pm \frac{5\pi}{6}$.
5. $2n\pi$.
6. $(2n+1)\pi$.
7. $2n\pi \pm \frac{\pi}{2}$.
8. $2n\pi$.
9. $2n\pi \pm \frac{\pi}{6}$, $2n\pi \pm \frac{5\pi}{6}$.
10. $2n\pi \pm \frac{\pi}{3}$, $2n\pi \pm \frac{2\pi}{3}$.

EXAMPLES 68. PAGE 137.

1. $n\pi + (-1)^n \frac{\pi}{6}$.
2. $n\pi + (-1)^n \frac{7\pi}{6}$.
3. $n\pi + (-1)^n \frac{\pi}{4}$.
4. $n\pi + (-1)^n \frac{5\pi}{4}$.
5. $n\pi$.
6. $n\pi + (-1)^n \frac{\pi}{2}$, i.e., $2n\pi + \frac{\pi}{2}$.
7. $n\pi + (-1)^n \frac{3\pi}{2}$, i.e., $2n\pi - \frac{\pi}{2}$.
8. $n\pi + (-1)^n \frac{\pi}{6}$, $n\pi + (-1)^n \frac{7\pi}{6}$.
9. $n\pi + (-1)^n \frac{\pi}{2}$, $n\pi + (-1)^n \frac{3\pi}{2}$.

EXAMPLES 69. PAGE 138.

1. $n\pi \pm \frac{\pi}{4}$.
2. $n\pi \pm \frac{\pi}{3}$.
3. $n\pi \pm \frac{\pi}{3}$.
4. $n\pi \pm \frac{\pi}{6}$.
5. $n\pi \pm \frac{\pi}{2}$.
6. $n\pi \pm \frac{\pi}{4}$.
7. $2n\pi + \frac{\pi}{6}$.
8. $2n\pi - \frac{\pi}{3}$.
9. $2n\pi + \frac{\pi}{4}$.
10. $2n\pi - \frac{\pi}{4}$.
11. $2n\pi + \frac{4\pi}{3}$.
12. $2n\pi + \frac{4\pi}{3}$.

EXAMPLES 70. PAGE 139.

1. $2n\pi + \frac{\pi}{2}$, $2n\pi$.
2. $2n\pi + \frac{\pi}{2}$, $(2n-1)\pi$.
3. $2n\pi$, $2n\pi - \frac{\pi}{2}$.
4. $2n\pi + \frac{\pi}{2}$, $2n\pi - \frac{\pi}{6}$.
5. $2n\pi + \frac{\pi}{2}$, $2n\pi - \frac{5\pi}{6}$.
6. $2n\pi$, $2n\pi - \frac{2\pi}{3}$.
7. $2n\pi + \frac{\pi}{4}$.
8. $2n\pi - \frac{\pi}{4}$.
9. $2n\pi - \frac{\pi}{12}$, $2n\pi - \frac{5\pi}{12}$.

EXAMPLES 71. PAGE 140.

1. $\frac{1}{3} \left\{ n\pi + (-1)^n \frac{\pi}{6} \right\}$.
2. $n\pi \pm \frac{\pi}{8}$.
3. $\frac{1}{6} (2n \pm 1) \pi$.
4. $\frac{1}{6} \left(n\pi + \frac{5\pi}{6} \right)$.
5. $2n\pi + (-1)^n \frac{5\pi}{2}$.
6. $4 \left(n\pi \pm \frac{\pi}{6} \right)$.
7. $\frac{1}{2} \left(n\pi \pm \frac{\pi}{6} \right)$.
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9. $n\pi + (-1)^n \frac{\pi}{2}$.
10. $n\pi + \frac{\pi}{4}$.
11. $n\pi \pm \frac{\pi}{4}$.
12. $n\pi$.
13. $2n\pi \pm \frac{\pi}{2}; n\pi + (-1)^n \frac{\pi}{3}$.
14. $n\pi + \frac{\pi}{4}$.
15. $2n\pi \pm \frac{\pi}{3}$.
16. $n\pi + \frac{\pi}{3}; n\pi + \frac{\pi}{4}$.
17. $n\pi; n\pi + (-1)^n \frac{\pi}{2}$.
18. $n\pi + \frac{5\pi}{12}; n\pi + \frac{\pi}{12}$.
19. $n\pi \pm \frac{\pi}{3}; n\pi \pm \frac{\pi}{4}$.
20. $2n\pi \pm \frac{\pi}{3}$.
21. $n\pi; n\pi \pm \frac{\pi}{3}$.
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34. $\frac{1}{2} \left(n\pi + \frac{\pi}{4} \right)$.

EXAMPLES 72. PAGE 141.

1. $\frac{\pi}{3}$.
2. 69·81 miles.
3. $1^\circ 3' 20''; 12^\circ 40'$.
4. $56\frac{1}{15}, 60, 63\frac{3}{15}$.
5. $\frac{1}{2}\sqrt{21}, -\frac{1}{2}\sqrt{21}$; or, $-\frac{1}{2}\sqrt{21}, \frac{1}{2}\sqrt{21}$.
12. $\bar{2}\cdot0087449$.
13. $1\cdot1580510$.
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